6. ENERGY PROPAGATION AND CAPILLARY EFFECT

We showed in the previous section that wave energy propagates at the group speed; ie. $\bar{\mathcal{F}} = \bar{E}.C_g$ This fact has interesting consequences. For example, consider generation of deep-water gravity waves on water starting from rest. The wave front, which separates the region with wave energy from still water region, travels at the group speed. The waves behind the front travel at the phase speed. In the case of deep water gravity waves, $C_g = 1/2\ C_p$, and therefore the waves will begin reach the wave front and disappear into the still water, as shown in the figure below. An imaginary surfer on a wave crest would reach the wave front and, if attempts to proceed further at the phase speed, would fall on the still water!

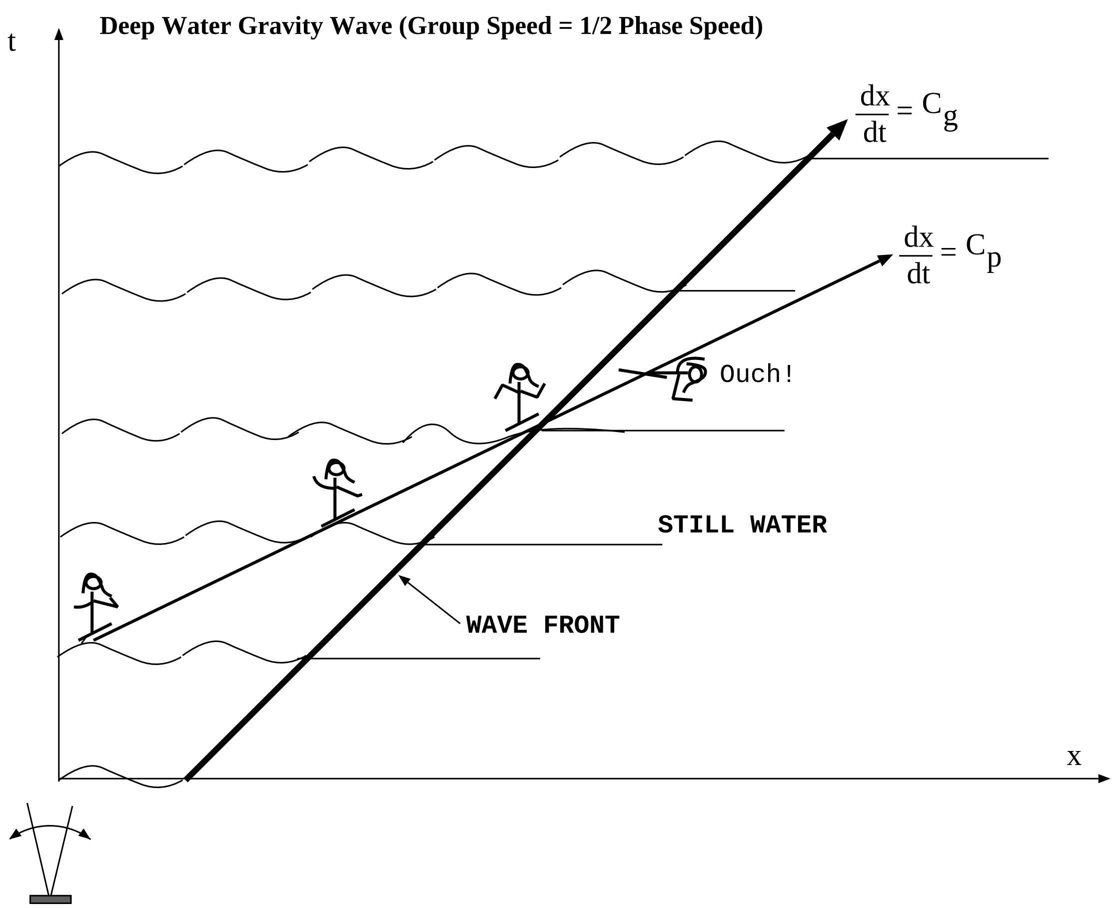

Fig 6-1. Generation of deep-water GRAVITY waves on water initially at rest

Effect of Surface Tension: Capillary Waves

Next, let us examine the effect of surface tension on surface waves. The formulation of the boundary-value problem is very similar to the case considered earlier without surface tension; the only condition that requires to be modified is the free-surface dynamic boundary condition. In the presence of surface tension, the gage pressure on the free surface will be nonzero and will be balanced by surface tension. In particular,

$$p - p_{atm} \equiv p(gage) = -s\kappa \text{ on the free surface}$$

where s denotes the surface-tension coefficient and κ the surface curvature (which is one over the radius of curvature). The expression for curvature, in terms of the free-surface elevation $\eta(x,t)$ is given by

$$\kappa \equiv \frac{1}{R} = \frac{-\partial^2\eta/\partial x^2}{\sqrt{1+(\partial\eta/\partial x)^2}}$$

For small-amplitude waves $(\partial\eta/\partial x)^2 << 1$, and therefore

$$p(gage) = -s\frac{\partial^2\eta}{\partial x^2} \text{ on the free surface} \tag{135}$$

Subsituting the above expression for gage pressure on the free surface $z = \eta(x,t)$ and linearizing, one obtains the following linearized free-surface dynamic condition including the effect of surface tension:

$$-\rho\frac{\partial\phi}{\partial t} + \rho g\eta = s\frac{\partial^2\eta}{\partial x^2} \text{ on } z = 0. \tag{136}$$

$$\tag{137}$$

One can obtain periodic solutions to the free-surface flow problem, including the effect of surface tension, in precisely the same manner as followed for the case without surface tension. In other words, one can solve the problem by the method of separation of variables and the integration constants from the boundary condition. Of particular interest here is the dispersion relation which is now given by

$$\sigma^2 = (gk + \frac{sk^3}{\rho}) \text{ Tanh}kh \tag{138}$$

The above relation shall be known as gravity-capillary wave dispersion relation. Note that for zero surface tension (ie. s =0), the relation reduces to the familiar $\sigma^2 = gk$ Tanhkh.

Gravity wave: If $gk >> \frac{sk^3}{\rho}$, then the dispersion relation can be reduced to

$$\sigma^2 = gk \text{ Tanh}kh$$

Waves satisfying this condition are the gravity waves. Note that the above inequality will be true for small wave number k, ie. for large wave length L. In other words, the effect of surface tension is negligible on large waves.

Capillary wave: On the other hand, if $gk << \frac{sk^3}{\rho}$, then the dispersion relation can be reduced to

$$\sigma^2 = \frac{sk^3}{\rho} \text{ Tanh}kh$$

These waves are called capillary waves. Note that the effect of surface tension is scaled by k^3 and gravity by k; in other words, the effect of surface tension is significant for waves of large k (ie. small waves). Of course, the effect of surface tension is significant when the gravitational acceleration is small as in outer space.

Properties of Deep-Water Gravity-Capillary Wave

Let us examine a particular case of deep-water waves ($kh \leq \pi$) in which case the dispersion relation becomes

$$\sigma^2 = gk + \frac{sk^3}{\rho}) \tag{139}$$

as Tanh$kh = 1$. Dividing by k^2 and noting that phase speed $C_p \equiv \sigma/k$, we obtain

$$\frac{\sigma^2}{k^2} \equiv C_p^2 = \frac{g}{k} + \frac{sk}{\rho} \tag{140}$$

Per above relation, the gravity tend to make longer waves travel faster in deep water (we already know this) while surface tension tend to make shorter waves travel faster. The combined effect results in a minumum for the wave speed on water, as illustrated in the figure below.

The minimum wave speed can be determined first by determining the k corresponding to the stationary point:

$$\frac{dC_p^2}{dk} = 0 \quad \rightarrow -\frac{g}{k^2} + \frac{s}{\rho} = 0 \quad \rightarrow k_{min} = \sqrt{\rho g/s}$$

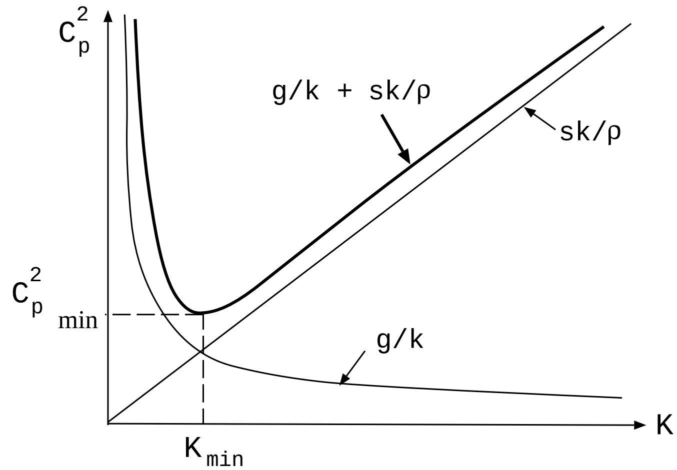

Fig 6-2. Effects of gravity and surface tension on wave speed in deep water

One can check that the k_{min} corresponds to a minimum of C_p. The $C_{p\ min}$ can now be computed by substituting k_{min} in the expression for C_p:

$$C^2_{p\ min} = 2\sqrt{sg/\rho}$$

In the case of clear air-water interface, s= 0.073 [N/m]. Assuming ρ to be 1000 [kg/m^3] and g to me 9.8 [m/s^2], we observe that the minimum wave speed on water is 0.23 [m/s]! If waves are found to propagate at a lower speed, it would mean that either the water density is different from 1000 [kg/m^3] or that the surface is contaminated making $s \neq 0.073$ [N/m].

Propagation of Deep-Water Capillary wave

Next, let us consider the case of purely capillary waves on deep water; ie, waves satisfying $kh \leq \pi$ and $sk^3/\rho >> gk$. In this case, the dispersion relation reduces to

$$\sigma^2 = \frac{sk^3}{\rho}$$

The wave (phase) speed for the wave is

$$C_p \equiv \sigma/k = \sqrt{sk/\rho}$$

while the group speed is

$$C_g \equiv \frac{d\sigma}{dk} = \frac{3}{2}\sqrt{sk/\rho}$$

In other words, interestingly, the group speed is greater than the phase speed in the case of deep-water capillary waves. You may recall that in the case of deep-water gravity waves $C_g = C_p/2$.

The generation of purely capillary deep-water waves starting from a quiescent (static) state is illustrated in the following figure. The wave front, which separates the region with waves and the still-water region, advances at the group speed (as wave energy propagates at group speed). The waves behind the front themselves propagate at the phase speed. As phase speed is less than the group speed, more and more number of waves will appear in the front even though the waves are generated from behind. As shown in the figure, a bug is on the first wave crest at some initial time and is flying forward at the wave's phase speed. It remains on the crest, but perhaps to its amazement, sees more and more waves popping in front of it even though the wave maker is behind it. The bug's world must be very confusing!

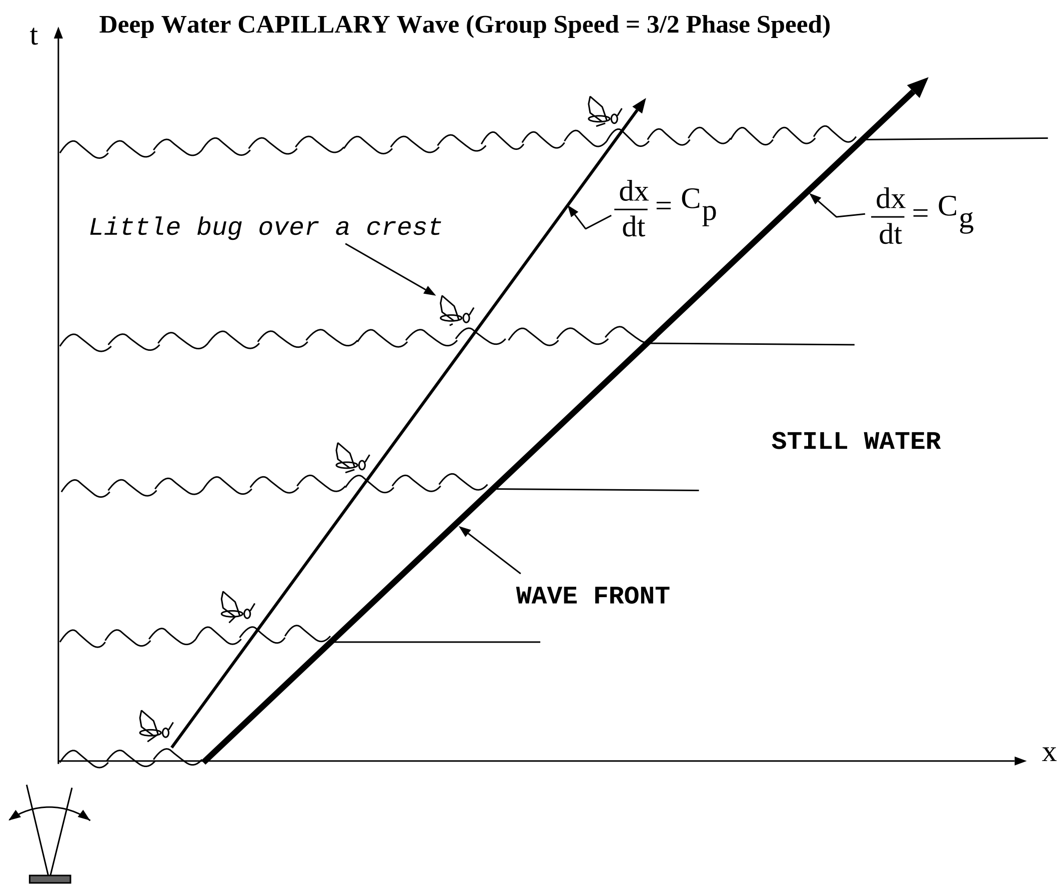

Fig 6-3. Generation of deep-water CAPILLARY waves on water initially at rest