4. Properties of Progressive Gravity Wave

In this chapter, we shall discuss kinematic and dynamic properties of ocean surface waves using a progressive wave solution obtained in the previous chapter.

Shallow- and Deep-Water Limits

We showed in the earlier chapter that wave frequency is related to wave number and water depth as

$$\sigma^2 = gk \ \text{Tanh}kh.$$

From mathematical tables one can note that $\text{Tanh}kh \rightarrow 1$ as $kh \rightarrow \infty$ and $\text{Tanh}kh \rightarrow kh$ as $kh \rightarrow 0$. The $kh \rightarrow \infty$ limit, which means that fluid depth is very large compared to wave length, is termed as *deep-water* or *short* wave limit. The $kh \rightarrow 0$ limit on the other hand, which means that fluid depth is very small compared to wave length, is called *shallow-water* or *long* wave limit. For practical calculations, one can take $kh \geq \pi$ to correspond to deep-water wave and $kh \leq \pi/10$ to correspond to *shallow-water* wave without loosing much of accuracy. For these cases, the above dispersion relation reduces to

$$\sigma^2 = gk, \ \text{for} \ kh \geq \pi \tag{81}$$

and

$$\sigma^2 = gk^2h, \ \text{for} \ kh \leq \frac{\pi}{10}. \tag{82}$$

For $\pi/10 < kh < \pi$, the so-called intermediate waves, these approximations are not valid, and one has to therefore use the full dispersion relation $\sigma^2 = gk \ \text{Tanh}kh$.

In the earlier chapter we showed that the phase speed of surface waves is given by

$$C_p = \frac{\sigma}{k} = \sqrt{\frac{g}{k} \ \text{Tanh}kh}.$$

For the case of deep-water (short) waves this relation reduces to

$$C_{p_{\text{deep}}} = \frac{\sigma}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{gL}{2\pi}}; \tag{83}$$

i.e., longer the waves, faster is their speed in deep water! In the case of shallow-water (or long) waves, the expression for phase speed becomes

$$C_{p_{\text{shallow}}} = \frac{\sigma}{k} = \sqrt{gh}; \tag{84}$$

i.e., wave speed decreases as the fluid depth decreases! In reality, this would mean that waves approaching a beach would tend to slow down, a wave property that is commonly observed. We shall investigate this property further as well as the steepening and breaking of waves in a later chapter that will deal with *transformation of waves in shallow waters.*

Fluid Particle Velocity in a Progressive Wave

Next, lets examine the motion of water particles due to progressive wave motion. For the examination, lets consider a progressive wave of the form

$$\begin{aligned} \eta &= \frac{H}{2}\sin(kx - \sigma t) \\ \phi &= \frac{H}{2}\frac{g}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\cos(kx - \sigma t) \end{aligned} \tag{85}$$

which we obtained in the earlier chapter. Spatial differentiation of ϕ yields the following expressions for particle velocity in the progressive wave:

$$\begin{aligned} u = & -\tfrac{\partial \phi}{\partial x} &= \frac{H}{2}\frac{gk}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\sin(kx - \sigma t) \\ w = & -\tfrac{\partial \phi}{\partial z} &= -\frac{H}{2}\frac{gk}{\sigma}\frac{\text{Sinh}k(h+z)}{\text{Cosh}kh}\cos(kx - \sigma t) \end{aligned} \tag{86}$$

From above expressions we note that w becomes zero on the bottom $z = -h$ as to be expected from the no-flux boundary condition. The horizontal component of velocity however does not vanish on the bottom, as this solution corresponds to wave motion in inviscid fluid. In reality, one could take $z = -h$ of inviscid analysis to be outer edge of bottom boundary layer and use the results in the bulk of the fluid away from the boundary layer. By comparing Eqns.(85 and 86), we also observe that the horizontal component of fluid velocity is in phase with the free-surface displacement η.

Fluid Particle Trajectory in a Progressive Wave

Let $\vec{R}(t) = (x(t),\ z(t))$ define trajectory of a fluid particle in the progressive wave given by Eqn.(85). By the definitions of particle velocity and velocity field, we have

$$\begin{aligned} u = & \tfrac{dx(t)}{dt} &= \frac{H}{2}\frac{gk}{\sigma}\frac{\text{Cosh}k(h+z(t))}{\text{Cosh}kh}\sin(kx(t) - \sigma t) \\ w = & \tfrac{dz(t)}{dt} &= -\frac{H}{2}\frac{gk}{\sigma}\frac{\text{Sinh}k(h+z(t))}{\text{Cosh}kh}\cos(kx(t) - \sigma t) \end{aligned} \tag{87}$$

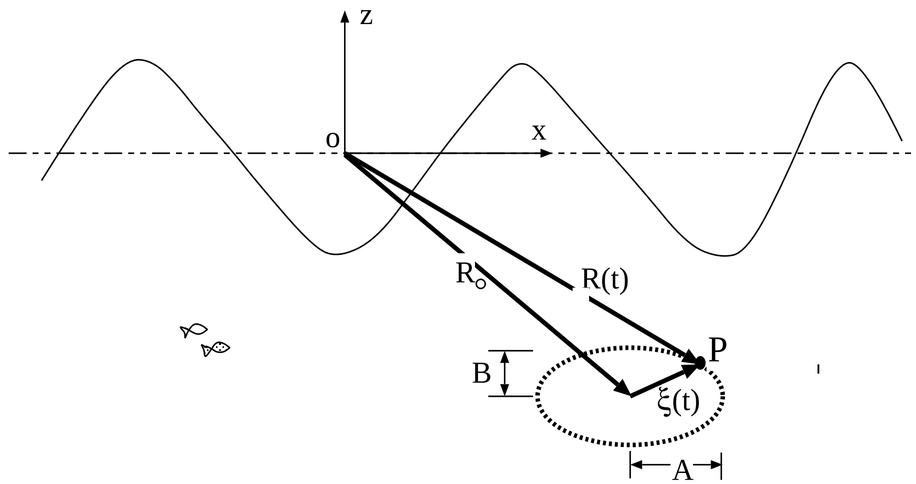

Fig 4-1. Particle trajectory in a progressive wave.

By integrating these equations with respect to time, one can obtain the equation of trajectory $x(t),\ z(t)$. This however not trivial as these equations are nonlinear with respect to unknowns $x(t),\ z(t)$. In the same spirit of linearization of free-surface conditions, we shall linearize these equations also. For this purpose, lets decompose the position vector $\vec{R}(t)$ into $\vec{R_o}$ and $\vec{\xi}(t)$ as shown in Fig. 4-1; *i.e.*,

$$\vec{R}(t) = \vec{R_o} + \vec{\xi}(t), \tag{88}$$

where $\vec{R_o}$ is the fixed equilibrium point of the particle and $\vec{\xi}(t)$ deviation of the particle from $\vec{R_o}$ because of the wave motion. We shall denote components of $\vec{R_o}$ as $(x_o,\, z_o)$ and of $\vec{\xi}(t)$ as $(\xi(t),\, \zeta(t))$. We shall assume $\vec{\xi}(t)$ to be very small compared to wave length and thus linearize the equation for particle trajectory.

Substituting the components of Eqn. (88) in Eqn. (87), we obtain

$$\begin{aligned} \frac{d\xi(t)}{dt} &= \frac{H}{2}\frac{gk}{\sigma}\frac{\mathrm{Cosh}k(h+\{z_o+\zeta(t)\})}{\mathrm{Cosh}kh}\sin(k\{x_o+\xi(t)\}-\sigma t) \\ \frac{d\zeta(t)}{dt} &= -\frac{H}{2}\frac{gk}{\sigma}\frac{\mathrm{Sinh}k(h+\{z_o+\zeta(t)\})}{\mathrm{Cosh}kh}\cos(k\{x_o+\xi(t)\}-\sigma t). \end{aligned} \tag{89}$$

Assuming particle deviation to be small compared to wavelength, *i.e.*, $k\zeta$ and $k\xi$ are negligibly small, one can linearize the above equation as

$$\begin{aligned} \frac{d\xi(t)}{dt} &= \frac{H}{2}\frac{gk}{\sigma}\frac{\mathrm{Cosh}k(h+z_o)}{\mathrm{Cosh}kh}\sin(kx_o-\sigma t) \\ \frac{d\zeta(t)}{dt} &= -\frac{H}{2}\frac{gk}{\sigma}\frac{\mathrm{Sinh}k(h+z_o)}{\mathrm{Cosh}kh}\cos(kx_o-\sigma t). \end{aligned} \tag{90}$$

These equations, when integrated with respect to time, give

$$\begin{aligned}
\xi(t) &= \frac{H}{2}\frac{gk}{\sigma^2}\frac{\mathrm{Cosh}k(h+z_o)}{\mathrm{Cosh}kh}\cos(kx_o-\sigma t)+C_1 \\
\zeta(t) &= \frac{H}{2}\frac{gk}{\sigma^2}\frac{\mathrm{Sinh}k(h+z_o)}{\mathrm{Cosh}kh}\sin(kx_o-\sigma t)+C_2,
\end{aligned} \tag{91}$$

where C_1 and C_2 are integration constants. These constants are however zero, since the wave deviation ξ and ζ are zero when there are no waves (*i.e.*, when wave height $H=0$). Thus, we have

$$\begin{aligned}
\xi(t) = \quad x(t)-x_o &= \frac{H}{2}\frac{gk}{\sigma^2}\frac{\mathrm{Cosh}k(h+z_o)}{\mathrm{Cosh}kh}\cos(kx_o-\sigma t) \\
\zeta(t) = \quad z(t)-z_o &= \frac{H}{2}\frac{gk}{\sigma^2}\frac{\mathrm{Sinh}k(h+z_o)}{\mathrm{Cosh}kh}\sin(kx_o-\sigma t).
\end{aligned} \tag{92}$$

Representing the coefficients of above equation in the form,

$$\begin{aligned}
A &\equiv \frac{H}{2}\frac{gk}{\sigma^2}\frac{\mathrm{Cosh}k(h+z_o)}{\mathrm{Cosh}kh} \\
B &\equiv \frac{H}{2}\frac{gk}{\sigma^2}\frac{\mathrm{Sinh}k(h+z_o)}{\mathrm{Cosh}kh},
\end{aligned} \tag{93}$$

we can write the equation for particle trajectory (Eqn. 92) as

$$\begin{aligned}
\xi(t) = \quad x(t)-x_o &= A\cos(kx_o-\sigma t) \\
\zeta(t) = \quad z(t)-z_o &= B\sin(kx_o-\sigma t).
\end{aligned} \tag{94}$$

When re-written as,

$$\left\{\frac{x-x_o}{A}\right\}^2+\left\{\frac{z-z_o}{B}\right\}^2=1, \tag{95}$$

one immediately recognizes that the trajectory of fluid particle is nothing but an ellipse with center at x_o, z_o, semi-major axis $A=\frac{H}{2}\frac{gk}{\sigma^2}\mathrm{Cosh}k(h+z_o)/\mathrm{Cosh}kh$, and semi-minor axis $B=\frac{H}{2}\frac{gk}{\sigma^2}\mathrm{Sinh}k(h+z_o)/\mathrm{Cosh}kh$. The trajectory of fluid particle in a progressive wave is closed, even though the wave energy itself propagates along x direction! This is however true only for the case of small-amplitude waves. In the case of large-amplitude waves, there will be a small drift of fluid with the wave, which is known as the the *Stokes Drift*. We shall discuss this nonlinear phenomenon in a later chapter.
