Reading Assignment: Index Notation

As it is not core to the subject of *Physical Oceanography*, the present topic is left as a reading assignment. If you have any questions, please see the instructor during the office hours.

The index notation is a convenient method to represent complex relations involving vectors and tensors. The indices i,j and k will be commonly used to represent components of a vector or a tensor. In the case of three-dimensional vectors or tensors, these indices take on values 1, 2 and 3. The indices, for the present purpose, will be denoted as subscripts. To be precise and consistent, one has to use the index as superscript, in the case of so-called contravariant tensors and as subscript, in the case of so-called covariant tensors. We need not worry about these subtlities for the purpose of this subject – in fact, this topic is somewhat of a digression – but it is worth reading a standard text on the subject such as the one *Tensor Calculus and Differential Geometry* by Sokolnikov. In fact, you will find that a summer self-reading of the above text to be, most likely, the most productive and memorable summer of your lifetime!

One of the basic rules in the index notation, is the Einsteins summation convention, per which a repeated index represents summation. For example, a vector $\vec{A}$ can be written as

$$\vec{A} = A_i \hat{e}_i$$

where A_i represents the componentof the vector in the direction i and $\hat{e}_i$ the unit (base) vector along the direction i. As the index i is repeated, the above equation represents

$$A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

or with respect to a Cartesian $oxyz$ coordinates, in the conventional form,

$$A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

Obviously writing $\vec{A}$ as $A_i \hat{e}_i$ is more compact than the latter traditional representations.

In the index notation, including the Einstein's summation convention, **addition of vectors** can be written as

$$\vec{A} + \vec{B} = (A_k + B_k) \hat{e}_k$$

The **dot (inner/scalar) product of two vectors** can be written as

$$\begin{aligned}\vec{A}\cdot\vec{B} &= A_i\hat{e}_i\cdot B_j\hat{e}_j\\ &= A_iB_j\ \hat{e}_i\cdot\hat{e}_j\end{aligned}$$

As the dot product of base vectors $\hat{e}_i$ and $\hat{e}_j$ is equal to 1 for i=j and equal to 0 for ineqj, we can introduce the following symbol, known as the *Kronecker delta* for the definition of the dot product. The Kronecker delta δ_{ij} is defined as follows:

$$\begin{aligned}\delta_{ij} &= 1 \text{ for } i=j\\ &= 0 \text{ for } i\neq j\end{aligned}$$

Therefore, the dot product of the unit vectors $\hat{e}_i$ and $\hat{e}_j$ is equal to δ_{ij}! The relation for dot (scalar) product of the two vectors $\vec{A}$ and $\vec{B}$ given above can be written as

$$\begin{aligned}\vec{A}\cdot\vec{B} &= A_iB_j\ \hat{e}_i\cdot\hat{e}_j\\ &= A_iB_j\delta_{ij}\\ &= A_iB_i \text{ (ignoring zero terms when } i\neq j).\end{aligned}$$

Thus in the index notation, $\vec{A}\cdot\vec{B}$ is simply A_iB_i. As index i is repeated, this actually represent $A_1B_1 + A_2B_2 + A_3B_3$ or in terms of $oxyz$ coordinates, $A_xB_x + A_yB_y + A_zB_z$.

The cross product of two vectors is represented using the permutation symbol ϵ_{ijk} which is defined as follows:

$$\begin{aligned}\epsilon_{ijk} &= +1, \text{ for even permutation of i,j and k; ie, for ijk = 123, 231, and 312.}\\ &= -1, \text{ for odd permutation of i,j and k; ie, for ijk = 132, 231, and 321.}\\ &= 0, \text{ if indices are repeated; ie, for ijk = 112, 221, 313, 223, 111, 222, 233, 311 etc}\end{aligned}$$

The **cross product** of two vectors is written in the index notation as

$$\vec{A}\times\vec{B} = \epsilon_{ijk}A_iB_j\hat{e}_k$$

which a very compact representation for a cross product. Expanding above, as indices i, j, k are repeated and ignoring zero terms (by the definition of ϵ_{ijk}), we can show that the right-hand side of the above is equal to

$$\epsilon_{ijk}A_iB_j\hat{e}_k = \hat{e}_1(A_2B_3 - A_3B_2) + \hat{e}_2(A_3B_1 - A_1B_3) + \hat{e}_3(A_1B_2 - A_2B_1)$$

and in terms of rectilinear $oxyz$ coordinates above is equal to

$$\epsilon_{ijk} A_i B_j \hat{e}_k = \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x)$$

which is the traditional, but lengthy, form of representing of $\vec{A} \times \vec{B}$.

Differential vector operations such as gradient, divergence and curl of a vector (or a tensor) can also be conveniently represented using the index notation. For illustration, we limit our discussion corresponding to only rectangular $oxyz$ coordinates. In the index notation, the **gradient** operator is represented as

$$\nabla = \frac{\partial}{\partial x_j} \hat{e}_j$$

As the index j is repeated, above expands to the familiar expression

$$\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

for the gradient operator defined in $oxyz$ coordinates.

The **divergence** of a vector can be written as

$$\nabla \cdot \vec{A} = \frac{\partial A_j}{\partial x_j}$$

As the index j is repeated (meaning summation) and as (x_1, x_2, x_3) correspond to xyz, the above index representation does correspond to the familiar expression for the divergence of a vector.

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

In the index notation, the **curl** of a vector can be denoted compactly as

$$\nabla \times \vec{A} = \epsilon_{ijk} \frac{\partial A_j}{\partial x_i} \hat{e}_k$$

Observing summation for repeated indices and definition of ϵ_{ijk} we can easily show that above corresponds to

$$\hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$$

The power and use of the index notation will become more apparent as we consider more involved and complex expressions and operations involving vectors and tensors. For example, the relation between stress vector $\vec{\tau}$ and stress tensor $\tilde{\sigma}$ is given by

$$\vec{\tau} = \tilde{\sigma} \cdot \hat{n}$$

In the index notation, it is written as

$$\vec{\tau} = \sigma_{ij} n_j \hat{e}_i$$

where n_j denotes the j-th component of the normal vector and $\hat{e}_i$ the unit vector along the direction i. With respect to rectangular oxyz coordinates, the above compact index representation corresponds to the lengthy expression

$$\hat{i}(\sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z) + \hat{j}(\sigma_{yx} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z) + \hat{k}(\sigma_{zx} n_x + \sigma_{zy} n_y + \sigma_{zz} n_z)$$

Recall, in the derivation of the equation of motion we have used a vector identity (without proof)

$$\nabla \cdot (\rho \vec{u})\vec{u} \equiv (\rho \vec{u} \cdot \nabla)\vec{u} + \vec{u} \nabla \cdot (\rho \vec{u})$$

The proof is straightforward with the use of index notation:

$$\nabla \cdot (\rho \vec{u})\vec{u} = \frac{\partial}{\partial x_i}(\rho u_j) u_k \hat{e}_k \delta_{ij}$$

Note the Kronecker delta δ_{ij} is because of the dot product operation in $\nabla \cdot (\rho \vec{u})\vec{u}$. Expanding above using product rule of differentiation

$$\begin{aligned}
\frac{\partial}{\partial x_i}(\rho u_j) u_k \hat{e}_k \delta_{ij} &= \{u_k \frac{\partial}{\partial x_i}(\rho u_j)\delta_{ij} + \rho u_j \delta_{ij} \frac{\partial}{\partial x_i} u_k\} \ \hat{e}_k \\
&= \{u_k \frac{\partial}{\partial x_i}(\rho u_i) + \rho u_i \frac{\partial}{\partial x_i} u_k\} \ \hat{e}_k \\
&= \{u_k \nabla \cdot (\rho \vec{u}) + (\rho \vec{u} \cdot \nabla) u_k\} \ \hat{e}_k \\
&= \vec{u} \nabla \cdot (\rho \vec{u}) + (\rho \vec{u} \cdot \nabla)\vec{u}
\end{aligned}$$

. .... proof!

We also claimed that $\nabla \cdot \tilde{\sigma}$ for an incompressible Newtonian fluid (with $\tilde{\sigma}$ given by

$$\sigma_{ij} = -p\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

where μ is viscosity coefficient) yields

$$-\nabla p + \mu \nabla^2 \vec{u}$$

The proof is as follows:

$$
\begin{aligned}
\nabla \cdot \tilde{\sigma} &= \frac{\partial}{\partial x_j}\sigma_{ij}\hat{e}_i \\
&= \frac{\partial}{\partial x_j}\{-p\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})\}\hat{e}_i \\
&= -\frac{\partial p}{\partial x_i}\hat{e}_i + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}\hat{e}_i + \mu\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}\hat{e}_i \\
&= -\frac{\partial p}{\partial x_i}\hat{e}_i + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}\hat{e}_i \ \ \text{(because } \frac{\partial u_j}{\partial x_j} = \nabla \cdot \vec{u} = 0 \text{ for incompressible fluid)} \\
&= -\nabla p + \mu\nabla^2\vec{u} \ \ \text{(because by summation convention, } \frac{\partial^2 u_i}{\partial x_j \partial x_j}\hat{e}_i \text{ represents the } \nabla^2\text{operator)}
\end{aligned}
$$

. ... proof!

The index notation is thus a powerful tool or method to not only represent complex relations and operations involving vectors or tensors but also for analysis and derivations. The notation is commonly used in many branches of mechanics, and in ocean engineering, in subjects such as *added-mass* theory. As said at the beginning, the text *Tensor Calculus and Differential Geometry* by Sokolnikov is a good reference and a useful subject.