11. INERTIA & ADDED-MASS FORCES ON A CYLINDER

Recall from the previous lecture that the dynamic pressure (gage) on the cylinder is given by

$$p_{cyl} \equiv p_{r=a,\theta} - p_O = \frac{\rho}{2}U^2(1 - 4\sin^2\theta) + \rho\frac{dU}{dt}(2a\ \cos\theta - l)$$

It was shown that the force in the direction of the flow associated with the steady pressure (drag force) is zero per potential flow solution, and that the drag force in a real fluid is an effect of viscosity. Next let us examine the in-line force (ie. the force in the direction of the flow) associated with the unsteady pressure term $\rho\frac{dU}{dt}(2a\ \cos\theta - l)$. Integrating the pressure, multiplied by $\cos\theta$ in order to get the force in the direction of the flow, we obtain:

$$F_I = \int_0^{2\pi} \rho\frac{dU}{dt}(2a\ \cos\theta - l)\ \cos\theta\ a d\theta = 2\pi\rho a^2\ \frac{dU}{dt}$$

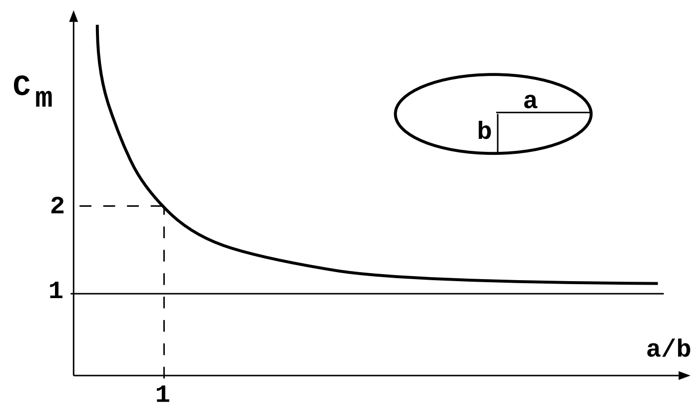

Fig 11-1. Inertia coefficients for elliptical cylinders.

This force which is due to the acceleration of the flow dU/dt is called the inertia force and denoted as F_I. For a general shape, the inertia force can be written as

$$F_I = C_M\rho\forall\frac{dU}{dt}$$

where C_M is called the inertia coefficient and $\forall$ denotes the volume of the body. For the case of circular cylinder, $\forall = \pi a^2$ (per unit length). Therefore, by inspecting the above two equation, we find that the inertia coefficient for circular cylinder is $C_M = 2$. The inertia coefficients for elliptical cylinders, with respect to the ratio of major and minor axes, is shown in Fig. 11-1. Please refer to the graph in the text book for exact quantitative values.

Added-Mass Force. Next, let us consider the problem of a circular cylinder of radius a moving in the positive x direction with a speed U(t) in otherwise static fluid. Remember that the case that has been discussed thus far is for stationary cylinder in a current of speed U(t). The potential flow equations governing the body-moving-in-otherwise-static-fluid problem are given by

$$\nabla^2\phi = 0 \quad \text{(Laplace equation)}$$

$$\frac{\partial \phi}{\partial r} = U \cos\theta \quad \text{on } r = a \quad \text{(where, a denotes the cylinder radius)}$$

$$\text{In the far field, ie. as } r \rightarrow \infty, \quad |\nabla \phi| = 0 \quad \text{(ie, cylinder motion not felt at } \infty)$$

As done before, one can solve the above equation by the method of separation of variables. And then using the Euler's integral, one can determine the dynamic pressure on the surface of the cylinder. As before, the pressure will have two sets of terms, one proportional to U^2 which is known as the steady pressure term, and the other proportional to dU/dt which is known as the unsteady pressure. Integrating the steady pressure (multiplied by -cosθ for the force component in the direction opposite to that of the body motion) one can find the drag force. As the problem formulation is based on the potential flow theory and as the pressure drag force is an effect of viscosity, the present result of drag force will be also zero (D'Alembert's paradox).

On the other hand, integrating the unsteady pressure (multiplied by -cosθ for the force component in the direction opposite to that of the body motion) one will that the hydrodynamic force acting in the direction opposite of the body motion is given by

$$F_a = \rho \pi a^2 \frac{dU}{dt}$$

which is known as the ADDED-MASS force. Note that the force on the stationary body in unsteady current is called INERTIA FORCE. They are related, however, as we will see in a moment. For a general shape, the added-mass force can be written as

$$F_a = \kappa_m \rho \forall \frac{dU}{dt}$$

where κ_m is called the added-mass coefficient and $\forall$ the volume of the body. As for the case of circular cylinder, the added-mass force is $F_a = \rho \pi a^2 \frac{dU}{dt}$, its added-mass coefficient is equal to one, $\kappa_m = 1$. One can show that the inertia coefficient C_m and added-mass coefficient κ_m are related as

$$C_m = \kappa_m + 1$$

An Illustrative Example: Let us consider a body under unsteady motion in the x direction with velocity U(t) in a fluid as shown in Fig. 11-2. Let the actual mass of the body be m. What is the force required to be applied ($F_{applied}$) to cause the body motion?

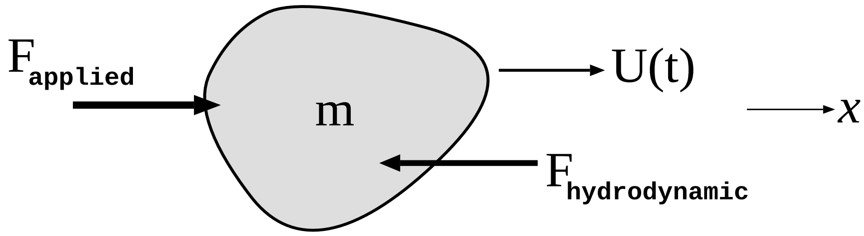

Fig 11-2. Body dynamics in a fluid

Answer: The hydrodynamic force acting on the body consists of drag and added-mass force components; ie.,

$$\begin{aligned} \mathrm{F}_{hyrodynamic} &= \text{Drag} + \text{Added-mass} \\ &= -C_D \frac{1}{2}\rho A\, U^2 \;-\; \kappa_m \rho \forall \frac{dU}{dt} \end{aligned}$$

where the minus signs indicate that the forces act in the negative x direction, ie., opposing the body motion. Also, note that C_D denotes the drag-force coefficient and A the projected area (refer to the chapter on Drag in Fluids I).

Per Newton's II law,

$$\begin{aligned} m\frac{dU}{dt} &= \mathrm{F}_{applied} \;+\; \mathrm{F}_{hydrodynamic} \\ &= \mathrm{F}_{applied} \;-C_D \frac{1}{2}\rho A\, U^2 \;-\; \kappa_m \rho \forall \frac{dU}{dt} \end{aligned}$$

which when re-written becomes

$$\mathrm{F}_{applied} = (m + \kappa_m \rho \forall)\, \frac{dU}{dt} + C_D \frac{1}{2}\rho A\, U^2$$

If the body were to be moved in vacuum the force required to be applied is only $m\frac{dU}{dt}$. In fluid, the applied force is required to overcome also the hydrodynamic force. A component of the hydrodynamic force which is proportional to acceleration dU/dt can be combined with the actual inertia force as

$$(m + \kappa_m \rho \forall)\, \frac{dU}{dt}$$

In other words, the hydrodynamic force term $\kappa_m \rho \forall$ seems to increase the mass from m to $(m + \kappa_m \rho \forall)$ and hence the term added-mass force to the hydrodynamic force that is equal to $\kappa_m \rho \forall \; dU/dt$. Note that if $U = constant$ (independent of time), the applied force has to only overcome the drag force.