9. WAVE FORCES

In this chapter, which is perhaps the most important from ocean engineering viewpoint, we shall discuss methods to determine wave forces acting on fixed and floating objects. Various approximate methods such as that based on the Froude-Krlov hypothesis, Morrison-Equation method, McCamy-Fuchs analysis etc will be presented. First, we shall begin with a simple case of two-dimensional unsteady flow past fixed circular cylinder in infinite fluid in order to bring forth the concepts of inertial and drag components of hydrodynamic force.

Two-Dimensional Flow Past a Fixed Circular Cylinder Consider an uniform but unsteady flow about a fixed cylinder of radius a as shown in Fig.9-1. The far-field velocity is $U(t)$ which is in the negative x (right to left) direction.

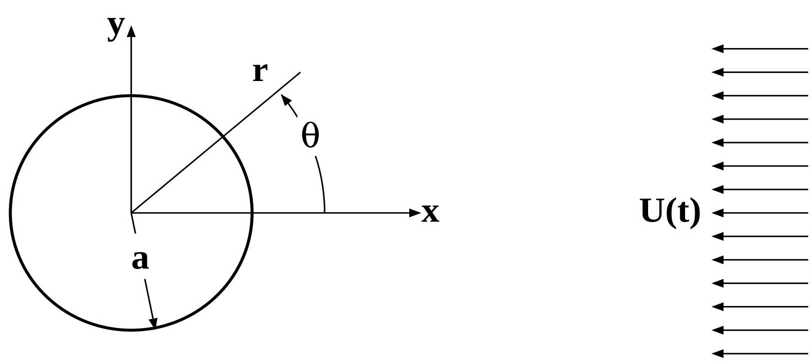

Fig 9-1. Flow past a cylinder

The problem is easier to solve using cylindrical polar coordinates (r, θ, z) which is related to the rectangular cartesian coordinates (x,y,z) as

$$x = r\ \cos\theta, \quad y = r\ \sin\theta, \quad z = z$$

Assuming the flow to be ideal (incompressible, homogenous and inviscid fluid, and irrotational flow), one can use the potential flow formulation to analyze the flow. In terms of the velocity potential $\phi = \phi(r, \theta, t)$, the radial and transverse components of fluid velocity field are given by

$$u_r = -\frac{\partial \phi}{\partial r}, \quad u_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

The two-dimensional Laplace equation in cylindrical coordinates is given by

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0 \tag{1}$$

On the cylinder, which is stationary, the no-flux of fluid gives

$$u_r \equiv -\frac{\partial\phi}{\partial r}=0 \text{ on } r=a \tag{2}$$

where a is the radius of the cylinder. As the body is closed, the effect of the body on the flow will vanish at infinity. We thus have the following far-field velocity condition for ϕ:

$$u_r \equiv -\frac{\partial\phi}{\partial r}=-U(t)\,\cos\theta \text{ as } r\to\infty \tag{3}$$

Separation of Variables. Using the method of separation of variables, one can easily solve the above problem. As per the method, the multi-variable function $\phi(r,\theta,t)$ is first written as a product of three single-variable functions; ie,

$$\phi(r,\theta,t)=R(r)\Theta(\theta)T(t)$$

Substituting it in the Laplace equation and multiplying by $r^2/R\Theta T$, we obtain

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=0$$

The first two-terms of the above equation depend only on r while the last term only on θ. As r and θ are independent, we can separate the above equation into two ordinary differential equations:

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}=+m^2$$

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=-m^2$$

where m is the separation constant which can be zero, imaginary or real. One can show that the cases of m being zero or imaginary do not satisfy the boundary condition. As we will see in a moment, only real m can satisfy the boundary conditions. Therefore, here, m is real constant.

Solution of Θ. The solution of the equation governing Θ is trivial:

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}-m^2 \quad\to\quad \Theta=A\,\sin m\theta+B\,\cos m\theta$$

Before proceeding further with the solution for $R(r)$, we may want to simplify the expression for Θ. As per the far-field condition

$$\frac{\partial\phi}{\partial r}=U\cos\theta$$

which means that the coefficient m in Θ must be equal to one; ie., $m=1$. Also, as the flow is inviscid the flow above the x-axis is mirror image of that below the x-axis; in other words, ϕ must be an <u>even</u> function of θ. Of the two terms in Θ, sine and cosine, only the cosine term is even with respect to θ. Thus the solution of Θ can therefore be reduced to

$$\Theta = B\ \cos\theta$$

Solution of R(r). Next, let us consider the solution of $R(r)$ given by

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} = +m^2, \text{ where as shown above } m = 1$$

Rewriting the above equation, we obtain

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - R = 0$$

Seeking solution of the form $R = \Sigma a_n r^n$ where a_n's are constants, we obtain the following characteristic equation for n:

$$\Sigma a_n r^n\{n(n-1) + n - 1\} = 0 \quad \rightarrow (n-1)(n+1) = 0 \quad \rightarrow \quad n = +1 \text{ and } -1.$$

Therfore,

$$R(r) = a_1 r + a_{-1}/r$$

where a_1 and a_{-1} are integration constants to be determined. The solution for ϕ is thus

$$\phi = R\Theta T = (a_1 r + \frac{a_{-1}}{r})B\ \cos\theta\ T$$

Absorbing the constant B into the constants a_1 and a_{-1}, we can write the solution as

$$\phi = (c_1 r + \frac{c_2}{r})\ \cos\theta\ T$$

where $c_1 \equiv a_1.B$ and $c_2 \equiv a_{-1}.B$. Note that the time function T is yet to be determined, as the Laplace equation does not contain time terms. T will depend on the time dependent far-field velocity $U(t)$.

Application of Boundary Conditions. The integration constants are determined using the boundary conditions. On the cylinder $r = a$ the radial component of velocity is zero, which gives

$$\frac{\partial\phi}{\partial r} = 0 \text{ on } r = a \quad \rightarrow \quad (c_1 - \frac{c_2}{a^2})\ \cos\theta\ T = 0 \quad \rightarrow c_2 = c_1 a^2$$

The solution of ϕ is now

$$\phi = c_1 T(r + \frac{a^2}{r})\ \cos\theta$$

Next, applying the far-field condition,

$$\frac{\partial\phi}{\partial r} = U(t)\ \cos\theta \quad \text{as } r \rightarrow \infty$$

we obtain

$$c_1T(1-\frac{a^2}{r^2})\ \cos\theta = c_1T\ \cos\theta \text{ (as r goes to infinity) } = U(t)\ \cos\theta\ \rightarrow c_1T = U(t)$$

The final solution of ϕ satisfying the Laplace equation and boundary conditions is thus

$$\phi = U(t)(r+\frac{a^2}{r})\ \cos\theta$$

Velocity Field. The components of the velocity field are given by

$$u_r \equiv -\frac{\partial\phi}{\partial r} = U(t)(1-\frac{a^2}{r^2})\cos\theta, \quad \text{and} \quad u_\theta \equiv -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U(t)(1+\frac{a^2}{r^2})\ \sin\theta$$

On the surface of the cylinder ($r = a$), we observe $u_r = 0$ (which in fact is the imposed no-flux condition) and $u_\theta = 2U(t)\ \sin\theta$. In other words, the fluid may slip along the surface of the cylinder, as the fluid is assumed to be inviscid. However, at $\theta = 0$ and π, on the surface of the cylinder, u_θ is also zero. The points ($r = a,\ \theta = 0$) and ($r = a,\ \theta = \pi$) on the cylinder where the total velocity is zero are known as the stagnation points. Instantaneous stream lines of the flow are sketched in the following figure (Fig. 9-2). Also show in the figure, are some illustrative streamlines in the case of real viscous flow in which case flow separation will occur and vortices shed in the wake creating the von Karman vortex street.

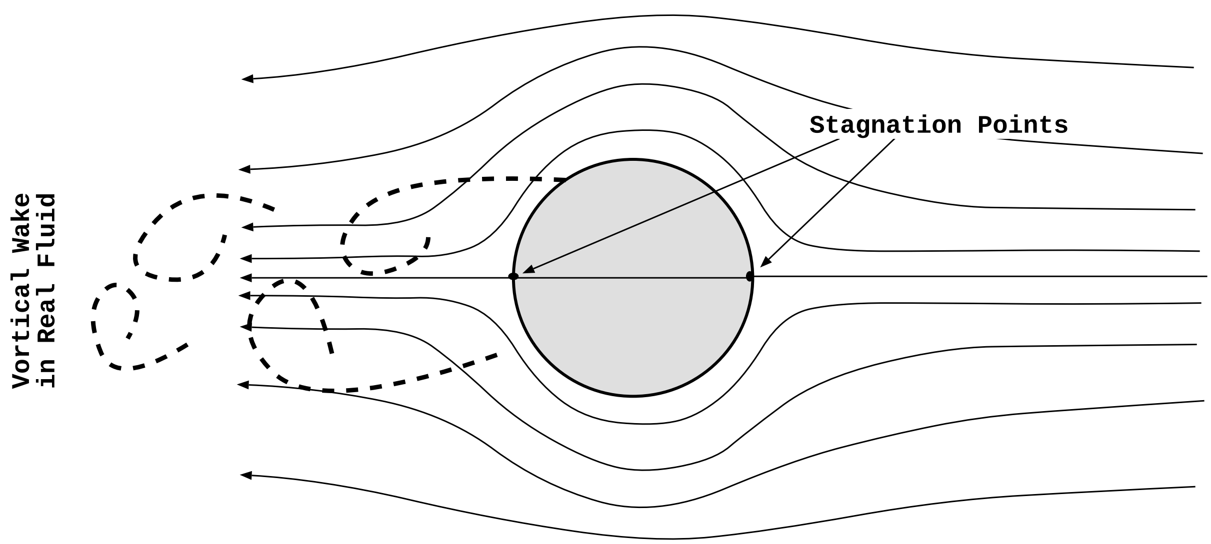

Fig 9-2. Instantaneous Streamlines: ideal vs. real flow