7. WAVE EVOLUTION & CONSERVATION OF WAVE NUMBER

Next, let us examine the temporal-spatial evolution of wave number k and frequency σ of a wave train generated by a transient event (eg., earthquake, storm, dropping a stone etc). To begin with, for simplicity, let us assume that the waves are propagating along the positive x direction. In other words, the phase function Θ of the waves can be written as

$$\Theta \equiv k(x,t)x - \sigma(x,t)t$$

Note that we now let k and σ to depend on x and time t. Differentiating Θ with respect to x and t, we get

$$\frac{\partial \Theta}{\partial x} = k + x\frac{\partial k}{\partial x} - t\frac{\partial \sigma}{\partial x},$$
$$\frac{\partial \Theta}{\partial t} = x\frac{\partial k}{\partial t} - t\frac{\partial \sigma}{\partial t} - \sigma$$

Under the assumption that the waves are SLOWLY VARYING, in other words that k and σ vary, but do not vary rapidly, with respect to x and t, one can approximate the above relations to

$$k \approx \frac{\partial \Theta}{\partial x},$$

$$\sigma \approx -\frac{\partial \Theta}{\partial t}$$

Note that the above relations are exact if k and σ are constants as in a monochromatic wave propagating in waters of uniform depth. Differentiating k with respect to time t and σ with respect to x and adding, we find

$$\frac{\partial k}{\partial t} + \frac{\partial \sigma}{\partial x} = \frac{\partial^2 \Theta}{\partial x \partial t} - \frac{\partial^2 \Theta}{\partial t \partial x} = 0$$

Thus we obtain an equation governing the evolution of k and σ as

$$\frac{\partial k}{\partial t} + \frac{\partial \sigma}{\partial x} = 0 \qquad (142)$$

Steady-State Waves: By definition, waves are said to be in *steady state*, if wave number k and and frequency σ are independent of time. For the case of steady-state waves, the above equation governing k and σ becomes

$$\frac{\partial k}{\partial t} + \frac{\partial \sigma}{\partial x} = 0 \quad \rightarrow \quad \frac{\partial \sigma}{\partial x} = 0 \;\; \text{as k is independent of t.}$$

In other words, in the case of steady-state waves, frequency σ is independent of space x also; ie., σ is simply a constant. Have in mind however that wavenumber k (and therefore wavelength L) can depend on x even in the case of steady-state waves.

Conservation of Wave Number: Recall that wavenumber k and frequency σ are related by the dispersion relation, and therefore the equation governing k and σ can be written as follows:

$$\frac{\partial k}{\partial t} + \frac{\partial \sigma}{\partial x} = \frac{\partial k}{\partial t} + \frac{\partial k}{\partial x}\frac{d\sigma}{dk} = 0$$

By definition, $d\sigma/dk$ is the group speed C_g. Therefore, the above equation can be written as

$$\frac{\partial k}{\partial t} + C_g \frac{\partial k}{\partial x} = 0 \tag{143}$$

which you may recall from an earlier course on *partial differential equations* is a model hyperbolic "wave" equation!. Here the equation will be referred to as the *wave number conservation equation*, the reason for which will become apparent in a moment.

The equation

$$\frac{\partial k}{\partial t} + C_g \frac{\partial k}{\partial x} = 0$$

subject to initial condition

$$k(x, t = 0) = f(x)$$

can be solved by a variety of methods including separation or transformation of variables and Fourier transform, as you may have done in the earlier course on engineering mathematics. Here, we will consider a graphical method known as the method of characteristics to analyze the above k-conservation equation. A *characteristic* is a line in the x, t plane on which the unknown k will remain constant. In other words, on the characteristic line,

$$k = \text{constant} \;\rightarrow\; dk = 0$$

As $k \equiv k(x,t)$, on the characteristic line

$$dk = 0 \;\rightarrow\; \frac{\partial k}{\partial x}dx + \frac{\partial k}{\partial t}dt = 0 \;\rightarrow\; \frac{dx}{dt} = -\frac{\partial k/\partial t}{\partial k/\partial x}$$

By the governing equation $\frac{\partial k}{\partial t} + C_g \frac{\partial k}{\partial x} = 0$,

$$-\frac{\partial k/\partial t}{\partial k/\partial x} = C_g$$

Therefore, the slope of the characteristic line (line on which k is constant) is given by

$$\frac{dx}{dt} = C_g$$

What does this solution mean? It means that on a frame moving at the group speed (ie. at $\frac{dx}{dt} = C_g$), the wave number will remain constant! In other words, the wave number is conserved at the group speed!! To illustrate the point, let us assume that one is asked to track a deep-water or intermediate-depth water wave of particular length (say 100 m) generated in a tsunami. The observer must move with the group speed corresponding to 100 m, so as to follow the wave of same 100 m length. As the observer is not moving at the phase speed, the wave will not appear stationary; the observer may be over the crest at one time and over the trough at another time. But, the wave length will appear the same for the observer who is moving at the group speed. Of

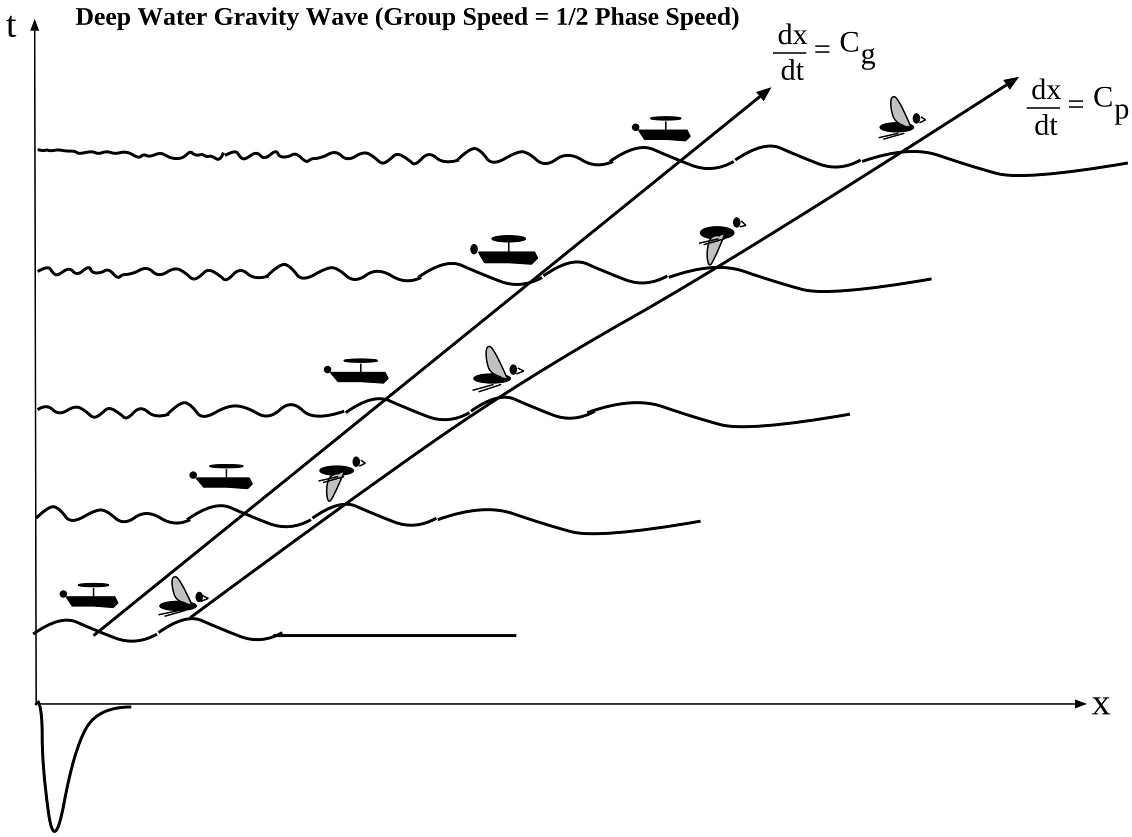

Fig 7-1. Conservation of Wave Number at Group Speed

course in the case when group speed is equal to the phase speed, as in the case of shallow-water gravity waves, both the phase and wave number will appear to be constant to the observer. The conservation of k and phase in deep or intermediate-depth water, where $C_p \neq C_g$ is illustrated in the sketch below.

In the above figure, gravity waves are generated by sort of a white-noise having a broad band of wave numbers, in deep water. As time progresses, notice the longer waves have travelled farther as, in the case of deep-water gravity waves longer waves travel faster. In other words the waves have DISPERSED. A helicopter is advancing at a group speed; an observer on the helicopter observes waves of constant wavenumber (or wavelength), but is not in phase with the wave. In other words, the helicopter moving at a group speed is over a crest at one time and over a trough at another time, but underneath it the wavelength remains constant. Also shown is a wild goose bird trying to fly over a crest at all times. The goose has to fly at the phase speed. Flying at phase speed, it remains on the crest but finds the wavelength of it to change in time (as the wavelength is conserved at group speed). The poor goose has to fly faster and faster, and perhaps give up chasing the wave at some point, as moving at the phase speed it finds the wavelength to keep increasing and therefore the wave speed also to keep on increasing.