2-5. Boundary Conditions for a General Wave Motion Problem

Lets consider a general wave motion problem as the one illustrated in the Fig. 2-3, below. The free surface (water-atmosphere interface) is denoted as $\mathcal{F}$, the bottom as $\mathcal{B}$, and the boundary of a body, if present, as $\mathcal{H}$. For defining such free-surface flow problem, it is customary to use a fixed coordinate system $oxyz$ with the xy plane on the calm water level (see figure below). The z axis points upward against the acceleration of gravity g. The unit normal vector on the boundaries, pointing out of the fluid, is denoted as $\hat{n}$. The velocity of the body $\mathcal{H}$ is shown as $\vec{U}$. We use $\eta(x, y, t)$ to represent the free-surface displacement about the calm level. The free surface is therefore defined by $F(x, y, z, t) = z - \eta(x, y, t) = 0$. Let the distance of the rigid bottom from the calm water level be denoted as $h(x, y)$, and therefore the equation of the bottom is $z - h(x, y) = 0$.

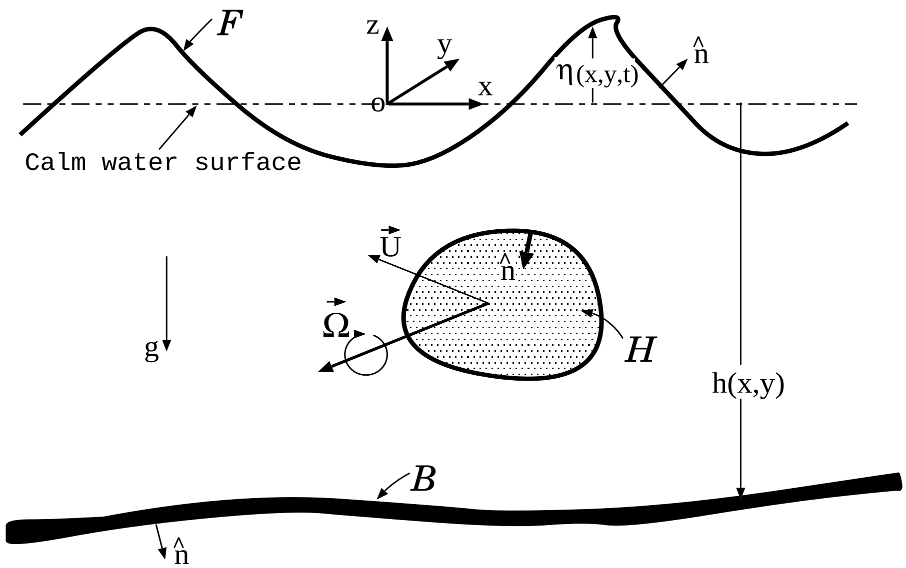

Fig 2-3. Free-surface flow: coordinate system and standard notations.

Bottom Boundary Condition

The bottom is assumed to be impermeable and rigid. In other words, there is no flux of fluid across the stationary bottom $\mathcal{B}$:

$$\vec{u} \cdot \hat{n} = 0 \text{ on } \mathcal{B}. \tag{1}$$

Since $\vec{u} = -\nabla\phi$, Eq. (1) becomes,

$$\vec{u} \cdot \hat{n} = -\nabla\phi \cdot \hat{n} = -\frac{\partial \phi}{\partial n} = 0, \text{ (or) } \frac{\partial \phi}{\partial n} = 0 \text{ on } \mathcal{B}. \tag{2}$$

The equation of the bottom being $z - h(x, y) = 0$, the unit normal vector $\hat{n}$ is given by

$$\hat{n} = \frac{\nabla[z - h(x,y)]}{|\nabla[z - h(x,y)]|} = \frac{-\frac{\partial h}{\partial x}\hat{i} - \frac{\partial h}{\partial y}\hat{j} + \hat{k}}{\sqrt{(\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial x})^2 + 1}} \tag{3}$$

Substituting this expression for $\hat{n}$ in Eq.(1), one can write the rigid, stationary bottom boundary condition as

$$\vec{u} \cdot \hat{n} = \vec{u} \cdot \frac{-\frac{\partial h}{\partial x}\hat{i} - \frac{\partial h}{\partial y}\hat{j} + \hat{k}}{\sqrt{(\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial x})^2 + 1}} = (u\hat{i} + v\hat{j} + w\hat{k}) \cdot \frac{-\frac{\partial h}{\partial x}\hat{i} - \frac{\partial h}{\partial y}\hat{j} + \hat{k}}{\sqrt{(\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial x})^2 + 1}} = 0 \tag{4}$$

Expanding and simplifying, the above form of the boundary condition reduces to

$$w - u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y} = 0 \text{ on } \mathcal{B}. \tag{5}$$

In terms of velocity potential ϕ we have

$$-\frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial h}{\partial y} = 0 \text{ on } \mathcal{B}. \tag{6}$$

Equations (1), (2), (5), and (6) all express the same no-flux condition in various forms. The appropriate form one would choose depends on the manner in which the bottom and flow are defined in a particular problem.

Body Boundary Condition

The rigid body $\mathcal{H}$ is under translation with velocity $\vec{U}$. For no-flux of fluid across $\mathcal{H}$, the normal component of fluid velocity must be equal to normal component of body velocity:

$$\vec{u} \cdot \hat{n} = \vec{U} \cdot \hat{n} \text{ on } \mathcal{H}. \tag{7}$$

(**Note:** For body also under rotation, the normal body velocity is $\vec{u} \cdot \hat{n} = (\vec{U} + \vec{\Omega} \times \vec{r}) \cdot \hat{n}$ on $\mathcal{H}$, where is $\vec{r}$ is position vector from the axis of rotation).
In terms of velocity potential ϕ, the above equation can be written as

$$\vec{u} \cdot \hat{n} = -\nabla\phi \cdot \hat{n} = -\frac{\partial \phi}{\partial n} = \vec{U} \cdot \hat{n} \text{ on } \mathcal{H}. \tag{8}$$

Note that in the case of forced body motion problem, $\vec{U}$ is known or specified. In the case of freely moving body, $\vec{U}$ has to be determined based on the dynamics of the body which is subjected to hydrodynamic force also. In this latter case, hydrodynamic and body-dynamic problems are coupled!

Free-Surface Boundary Condition

As noted earlier, let $F(x, y, z, t) = z - \eta(x, y, t) = 0$ be the equation of the surface, with $\eta(x, y, t)$ representing free-surface deformation (or elevation) about the calm surface. Note that in the wave motion problem, η is not known *a priori* but is part of the solution. We therefore need two boundary conditions on the free surface, one of which we shall obtain based on the kinematics and the other on the dynamics considerations.

Free-Surface Kinematic Condition

A free surface is a *material surface*. In other words, fluid particles on the free surface cannot leave the surface, even though they may slide along the surface. This means that the normal component of fluid velocity on the free surface must be the same as the normal velocity of the surface (lets call it V_n):

$$\vec{u} \cdot \hat{n} = V_n \text{ on } \mathcal{F}. \tag{9}$$

You may recall from calculus of curves and surfaces, that normal velocity of a surface $F(x, y, z, t) = z - \eta(x, y, t) = 0$ is given by

$$V_n = \frac{-\frac{\partial F}{\partial t}}{|\nabla F|}. \tag{10}$$

And the unit normal vector $\hat{n}$ is given by

$$\hat{n} = \frac{\nabla F}{|\nabla F|}. \tag{11}$$

Substituting Eqs. (10) and (11) in Eq. (9), we obtain

$$\vec{u} \cdot \hat{n} = \vec{u} \cdot \frac{\nabla F}{|\nabla F|} = V_n = \frac{-\frac{\partial F}{\partial t}}{|\nabla F|} \text{ on } \mathcal{F}. \tag{12}$$

By cancelling the common terms on the left- and right-hand sides of the equation and rearranging, we get

$$\frac{\partial F}{\partial t} + \vec{u} \cdot \nabla F \equiv \frac{DF}{Dt} = 0 \text{ on } \mathcal{F}. \tag{13}$$

Upon expansion using $F(x, y, z, t) = z - \eta(x, y, t)$, the above equation becomes

$$-\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} - v\frac{\partial \eta}{\partial y} + w = 0 \text{ on } \mathcal{F}, \tag{14}$$

or in terms of velocity potential ϕ,

$$-\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} - \frac{\partial \phi}{\partial x} = 0 \text{ on } \mathcal{F}. \tag{15}$$

This equation which is obtained based purely on the free-surface kinematics is called the free-surface kinematic condition. Note that this equation is to be satisfied on $z = \eta(x, y, t)$ a surface not known *a priori*. Also, the quadratic terms of unknowns ϕ and η in this condition makes it nonlinear.

Free-Surface Dynamic Condition (without surface tension)

In the absence of surface tension, the pressure is continuous across the free surface. In other words, the fluid pressure is the same as the atmospheric pressure (or the gage pressure zero) on $\mathcal{F}$. With gage pressure being zero, the Euler's integral (see Eq. 22 on p.7) becomes

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + g\eta = 0 \text{ on } z = \eta. \tag{16}$$

This equation is called the free-surface dynamic condition. Like the free-surface kinematic condition, this is also nonlinear and has to be satisfied on the surface $z = \eta(x, y, t)$ which itself is an unknown. Because of free-surface nonlinearity, analysis of wave motion problem even with the potential-flow idealization is extremely difficult.

Before examining free-surface flows subject to exact free-surface conditions derived above, lets consider cases where linearization of the conditions is justifiable. Linearization of free-surface conditions is especially justifiable for small-amplitude waves, in which the wave amplitude is assumed insignificant compared to the wave length. Linearization of free-surface conditions is presented next.

Linearized Free-Surface Boundary Conditions

We shall assume that wave motion is "small". In particular, we shall assume that wave deformation and fluid velocity to be so small that we can ignore the products of η, ϕ, and their derivatives. Without such product terms, the free-surface kinematic and dynamic conditions (Eqs. 15, 16) become

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial z} = 0 \text{ on } z = \eta \tag{17}$$

$$-\frac{\partial \phi}{\partial t} + g\eta = 0 \text{ on } z = \eta. \tag{18}$$

The quantities $\frac{\partial \phi}{\partial z}$ and $\frac{\partial \phi}{\partial t}$ appearing in these equations have to be still evaluated on unknown $z = \eta$. One can surmount this difficulty by Taylor expanding these terms about $z = 0$ (calm surface) and neglecting products of ϕ and η terms as done above, and get

$$\frac{\partial \phi}{\partial z}(x, t, z = \eta, t) = \frac{\partial \phi}{\partial z}(x, t, z = 0, t) + \frac{\eta}{2!}\frac{\partial^2 \phi}{\partial z^2}(x, t, z = 0, t) + ... \tag{19}$$

$$\approx \frac{\partial \phi}{\partial z}(x, t, z = 0, t) \tag{20}$$

$$\frac{\partial \phi}{\partial t}(x, t, z = \eta, t) = \frac{\partial \phi}{\partial t}(x, t, z = 0, t) + \frac{\eta}{2!}\frac{\partial^2 \phi}{\partial z \partial t}(x, t, z = 0, t) + ... \tag{21}$$

$$\approx \frac{\partial \phi}{\partial t}(x, t, z = 0, t). \tag{22}$$

The free-surface conditions to the leading order thus become

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial z} = 0 \quad \text{and} \quad -\frac{\partial \phi}{\partial t} + g\eta = 0 \text{ on } z = 0. \tag{23}$$

These are the linearized free-surface conditions, which need to be satisfied merely on the calm surface $z = 0$. Also note one can eliminate η by combining time-derivative of these equations, and get

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \text{ on } z = 0. \tag{24}$$

This equation for ϕ is known as the *combined linear free-surface condition.* It is important to have in mind that solutions obtained based on these linear free-surface conditions will be valid only for *small-amplitude waves.*

Lateral Boundary Condition

Finally, we have to specify certain boundary conditions in the lateral direction as well in order to obtain unique solutions for waves in *open* ocean. Depending on the nature of the wave motion, we shall use periodicity condition, Sommerfeld condition, etc. We shall elaborate these when we solve specific free-surface problems later. On the other hand, for wave motion in a *closed domain*, where the lateral boundaries correspond to stationary or moving rigid walls, one has to only apply the well-known no-flux condition.