EOC4422 OCEAN WAVE MECHANICS FALL 2005

1. Introduction

In this course, which is perhaps the most important subject in ocean engineering, we shall examine (i) basic kinematic properties of surface waves, (ii) transformation of waves as they approach land from deep ocean, (iii) generation of waves in a wave tank (iv) wave forces on fixed and floating ocean structures, (v) response of freely-floating and moored bodies to surface waves, (vi) spectral representation of sea states and body response, (vii) long waves in shallow water, (viii) large-amplitude waves and wave breaking, and time permitting topics such as, (viii) internal waves and (ix) generation of waves by wind. As in most branches of mechanics, we shall tackle these problems by first developing an appropriate mathematical model. The equations governing wave motion will be derived based on the familiar *principles of classical mechanics*: *viz.*, principle of conservation of mass, principle of balance of momentum, and the principle of work and energy. The length scales of interest in this subject is such that we can assume that water medium to be a *continuum* and quantify its mass in terms of density. The velocity scale encountered in a typical ocean wave is such that we can also assume water to be an incompressible fluid. The effect of viscosity on wave motion is important in some cases, rigorous treatment of which is a formidable task and is beyond the scope of this course. For the most part in this course, we will ignore the effect of viscosity and assume water to be *inviscid*, and consequently *irrotational*. Under these assumptions, the mathematical models of ocean waves become simpler to analyze. Despite the assumptions, as we shall see, the model can predict a wide range of wave motions with an accuracy that is reasonable for most engineering applications. In cases where *inviscid-flow* model is inadequate, we shall incorporate the effect of viscosity into the model, in an empirical manner. It is worthwhile to add that physics of waves is not yet fully understood, and perhaps the present course would motivate you to pursue further study and research in water-wave mechanics which – as you will find out during the course – is also the most beautiful and exciting subject in ocean engineering!

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2. Equations Governing Ocean Wave Flows

Starting from the basics of mechanics, we shall derive the equations necessary for solving a wave motion problem. The following integral identities of Gauss and the control-volume equation will be used for the derivations:

Gauss Theorems:

Let $\vec{A}$ be a any vector quantity defined in a region Ω (a control volume) bounded by surface S. Let $\hat{n}$ be unit normal outward vector on S (see Fig. 2-1). Then,

$$\int_S \vec{A} \cdot \hat{n} \, dS \equiv \int_\Omega \nabla \cdot \vec{A} \, d\Omega, \tag{1}$$

where

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

denotes the gradient operator. Also recall $\nabla \cdot \vec{A}$ is the *divergence* of the vector $\vec{A}$.

In the case of a scalar quantity a, an analogous integral identity exists which is given by:

$$\int_S a \, \hat{n} \, dS \equiv \int_\Omega \nabla a \, d\Omega. \tag{2}$$

Using above integral identities one can transform volume integrals to surface integrals, and *vice versa*, as we would see shortly in the derivation of governing equations for fluid flows.

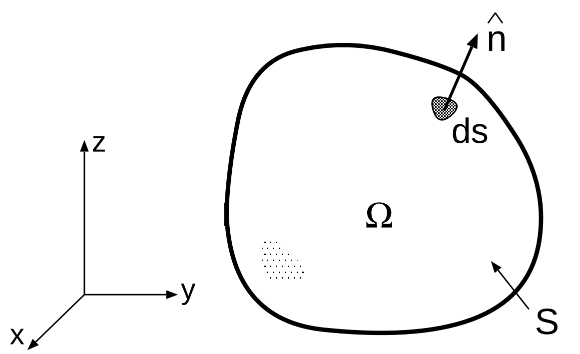

Fig 2-1. A region Ω of fluid bounded by surface S

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Control Volume Equation:

Let B denote an *extensive property* (one that depends on the amount of matter) and β the corresponding *intensive property* (*i.e.*, extensive property per unit mass). Let Ω denote a control volume and S its boundary (as in Fig. 2-1). Let $\vec{u}$ be fluid velocity and ρ fluid density. You may recall from Fluid Mechanics I course the following control-volume equation for the time rate of change of the extensive property of a *system*, B_{system}:

$$\frac{dB_{system}}{dt} = \frac{d}{dt}\int_\Omega \rho\beta d\Omega + \int_S \rho\beta\vec{u} \cdot \hat{n} \, dS \tag{3}$$

A somewhat analogous equation is called the *Leibnitz theorem* in mathematics and *Reynolds transport theorem* in mechanics. Using the above equation with B corresponding to mass, linear momentum etc, and in conjunction with *the principles of classical mechanics*, we shall derive the necessary equations for solving a fluid flow.

2-1. Equation of Continuity

Lets consider a *fixed* region Ω (as in Fig. 2-1) of fluid whose density is given by $\rho \equiv \rho(x, y, z, t)$ and velocity by $\vec{u} \equiv \vec{u}(x, y, z, t) \equiv (u, v, w)$. Let S denote the boundary of Ω. The instantaneous mass of fluid in the region is

$$m = \int_\Omega \rho \; d\Omega.$$

The mass rate of flow out of Ω is given by

$$\rho Q = \int_S \rho \vec{u} \cdot \hat{n} \; dS.$$

As mass is neither created nor destroyed, the rate of change of mass in Ω must be equal to the mass rate of flow into Ω: *i.e.*,

$$\frac{d}{dt} \int_\Omega \rho \; d\Omega + \int_S \rho \vec{u} \cdot \hat{n} \; dS = 0 \tag{4}$$

One can obtain this equation (4) directly, by applying the control-volume equation with B corresponding to mass m. As by definition of a system, and letting Ω to be fixed, we obtain

$$\begin{aligned} \frac{dm_{system}}{dt} = 0 \;\; &= \;\; \frac{d}{dt} \int_\Omega \rho \; d\Omega + \int_S \rho \vec{u} \cdot \hat{n} \; dS \\ &= \;\; \int_\Omega \frac{\partial \rho}{\partial t} \; d\Omega + \int_S \rho \vec{u} \cdot \hat{n} \; dS \end{aligned}$$

Note that the intensive property β corresponding to mass is simply unity.

Using the Gauss integral identity (with $\vec{A}$ corresponding to $\rho\vec{u}$), we can transform Eqn (4) as

$$\int_\Omega \frac{\partial}{\partial t} \rho \; d\Omega + \int_\Omega \nabla \cdot (\rho \vec{u}) \; d\Omega = \int_\Omega [\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \vec{u})] d\Omega = 0.$$

As the above integral (known as the **integral form of conservation of mass**) vanishes, irrespective of the size or geometry of Ω, the integrand must itself be zero: *i.e.*,

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \vec{u}) = 0,$$

which is called the *equation of continuity*. By expanding the $\nabla \cdot (\rho\vec{u})$ and rearranging terms, we can cast the above equation in a familiar form:

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \vec{u}) = \frac{\partial \rho}{\partial t} + [\vec{u} \nabla \rho + \rho \nabla \cdot \vec{u}] = [\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho] + \rho \nabla \cdot \vec{u} = 0.$$

Recognizing that $\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho$ is nothing but the material derivative of density $\frac{D\rho}{Dt}$, we can rewrite the above equation of continuity as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0. \tag{5}$$

Incompressible Fluid: Incompressible fluid is one in which the density of fluid particle remains constant. In other words, $\frac{D\rho}{Dt} = 0$ in an incompressible fluid. The equation of continuity given above therefore becomes

$$\nabla \cdot \vec{u} = 0 \tag{6}$$

for an incompressible fluid. In other words, in an incompressible fluid the divergence of velocity field is zero.

2-2. Balance of Linear Momentum in Inviscid Fluid: Euler's Equation

Next, we shall derive the equation of motion for inviscid fluid based on the Newton's second law of motion. Again, lets consider an arbitrary region Ω of fluid bounded by surface S. The external forces acting on the fluid in Ω are (i) due to pressure, p and (ii) due to gravity g acting in the negative z direction (Fig. 2-2). Lets denote the velocity field by $\vec{u}$.

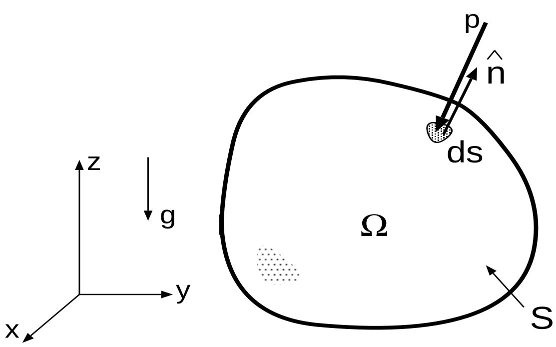

Fig 2-2. Force of pressure and gravity on fluid in region Ω bounded by surface S

The external force $\vec{F}$, due to gravity and pressure, is given by

$$\vec{F} = -\hat{k} \int_{\Omega} \rho g \; d\Omega - \int_{S} p\hat{n} \; dS.$$

Note that pressure is a surface stress and that it acts in the direction opposite of $\hat{n}$ (and hence the negative sign for the surface integral of pressure). Using the control-volume equation (3) with B

corresponding to linear momentum $m\vec{u}$, one can express the rate of change of linear momentum of a system of fluid which occupies Ω at the instant of time t as:

$$\frac{d}{dt}(m\vec{u})_{sys} \equiv \int_{\Omega} \frac{\partial \rho \vec{u}}{\partial t}\ d\Omega + \int_{S} (\rho\vec{u})\vec{u}\cdot\hat{n}\ dS.$$

As per Newton's second law of motion (rate of change of linear momentum of a system is equal to the sum of the external forces acting on the system), we therefore have

$$\int_{\Omega} \frac{\partial \rho \vec{u}}{\partial t}\ d\Omega + \int_{S} (\rho\vec{u})\vec{u}\cdot\hat{n}\ dS = -\hat{k}\ \int_{\Omega} \rho g\ d\Omega - \int_{S} p\hat{n}\ dS.$$

Using the Gauss integral identities (Eqns. 1 and 2), the surface integrals in the above equation can be transformed to volume integrals:

$$\int_{\Omega} \frac{\partial \rho \vec{u}}{\partial t}\ d\Omega + \int_{\Omega} \nabla\cdot[(\rho\vec{u})\vec{u}]\ d\Omega = -\hat{k}\ \int_{\Omega} \rho g\ d\Omega - \int_{\Omega} \nabla p\ d\Omega.$$

As the above integral relation (known as the integral form of the conservation of linear momentum) is true for any Ω, we can simply equate the integrals:

$$\frac{\partial \rho \vec{u}}{\partial t} + \nabla\cdot[(\rho\vec{u})\vec{u}] = -\rho g\hat{k} - \nabla p \tag{7}$$

As by chain rules,

$$\frac{\partial \rho \vec{u}}{\partial t} \equiv \rho\frac{\partial \vec{u}}{\partial t} + \vec{u}\frac{\partial \rho}{\partial t}$$

and

$$\nabla\cdot[(\rho\vec{u})\vec{u}] \equiv (\rho\vec{u})\cdot\nabla\vec{u} + \vec{u}\nabla\cdot(\rho\vec{u}),$$

we can write Eqn.(7) as

$$\rho\frac{\partial \vec{u}}{\partial t} + \vec{u}\frac{\partial \rho}{\partial t} + \nabla\cdot[(\rho\vec{u})\vec{u}] \equiv (\rho\vec{u})\cdot\nabla\vec{u} + \vec{u}\nabla\cdot(\rho\vec{u}) = -\rho g\hat{k} - \nabla p.$$

Grouping first and third terms on the left-hand side together and second and fourth terms together, we can rewrite the above equation as

$$\vec{u}[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u})] + \rho[\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}] = -\rho g\hat{k} - \nabla p. \tag{8}$$

By conservation of mass

$$[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u})] = \frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u} = 0.$$

Therefore, Eqn.(8) becomes

$$\rho(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}) = -\rho g\hat{k} - \nabla p \tag{9}$$

which is known as the **Euler's equation**.

Other Forms of Euler's Equations

In literature, one might find the Euler's equations presented in different forms. We list these forms below.

Recognizing that $\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \equiv \frac{D\vec{u}}{Dt}$, we can write the Euler's equation as:

$$\rho \frac{D\vec{u}}{Dt} = -\rho g \hat{k} - \nabla p. \tag{10}$$

By a vector identity,

$$\vec{u} \cdot \nabla \vec{u} \equiv \frac{1}{2} \nabla |\vec{u}|^2 - \vec{u} \times (\nabla \times \vec{u})$$

Therefore the Euler's equation (9) can be written as

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \frac{1}{2} \nabla |\vec{u}|^2 - \vec{u} \times \nabla \times \vec{u} \right) = -\rho g \hat{k} - \nabla p. \tag{11}$$

Note that $\nabla \times \vec{u} \equiv \vec{\omega}$ is called fluid **vorticity**. In terms of $\vec{\omega}$, the Euler's equation above can be expressed as

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \frac{1}{2} \nabla |\vec{u}|^2 - \vec{u} \times \vec{\omega} \right) = -\rho g \hat{k} - \nabla p. \tag{12}$$

In the case of constant-density fluid, the gravity term can be written as

$$-\rho g \hat{k} = -\nabla \rho g z.$$

Thus the right hand side of the Euler's equation can be combined as

$$-\rho g \hat{k} - \nabla p = -\nabla \rho g z - \nabla p = -\nabla (p + \rho g z)$$

For reasons that would become clear later, the quantity $p + \rho g z$ is called the **dynamic pressure**.

To summarize, the Euler's equation for constant density (incompressible, homogeneous) fluid can be in the following forms:

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla (p + \rho g z). \tag{13}$$

$$\rho \frac{D\vec{u}}{Dt} = -\nabla (p + \rho g z). \tag{14}$$

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \frac{1}{2} \nabla |\vec{u}|^2 - \vec{u} \times \vec{\omega} \right) = -\nabla (p + \rho g z). \tag{15}$$

$$\tag{16}$$

2-3. Potential Flow: Laplace Equation

In the absence of viscosity, the flow would be irrotational (*i.e.*, $\vec{\omega} \equiv \nabla \times \vec{u} = 0$). One can formally make this claim based on what is known as the Kelvin's theorem, something you would study later in an advanced course in hydrodynamics. And in the case of irrotational flow, the velocity field can be expressed as a negative (or positive) gradient of a scalar field: *i.e.*,

$$\nabla \times \vec{u} = 0 \;\; <=> \vec{u} = -\nabla\phi, \tag{17}$$

where ϕ is called the velocity potential. Hence, an irrotational flow is also called as a **potential flow**. An incompressible and irrotational flow is also referred to as the **ideal flow**. Note that we have expressed velocity of ideal flow as **negative gradient of** ϕ in Eqn.(17) following the the text of Dean and Dalrymple. In other books, one might find that velocity of ideal flow expressed as **positive gradient of** ϕ. We shall stick to the convention used in the text book of Dean and Dalrymple.

In terms of cartesian coordinate system xyz, the velocity components of a potential flow can be determined as

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}, \quad w = -\frac{\partial\phi}{\partial z}. \tag{18}$$

Thus knowing the velocity potential ϕ, one can determine components of velocity vector, simply by differentiating ϕ. We shall next derive the equations for determining ϕ.

In the case of incompressible-fluid flow, we have shown that $\nabla \cdot \vec{u} = 0$ (see Eqn. 6). If the flow is also irrotational, $\vec{u} = -\nabla\phi$. Substituting $-\nabla\phi$ for $\vec{u}$, we get

$$\nabla \cdot \vec{u} = -\nabla \cdot \nabla\phi = -\nabla^2\phi = 0. \tag{19}$$

The operator $\nabla \cdot \nabla \equiv \nabla^2$ is called the **Laplacian** operator. With respect to cartesian coordinate xyz the Laplacian operator is given by

$$\nabla \cdot \nabla \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{20}$$

is called the **Laplace equation**.

2-4. Euler's Integral (or) Bernoulli's Equation for Potential Flow

For an irrotational flow ($\vec{\omega} = 0$), Euler's equation (21) becomes

$$\rho(\frac{\partial\vec{u}}{\partial t} + \frac{1}{2}\nabla|\vec{u}|^2 = -\nabla(p + \rho g z). \tag{21}$$

Using $\vec{u} = -\nabla\phi$ in the above equation, we get

$$\rho(-\frac{\partial\nabla\phi}{\partial t} + \frac{1}{2}\nabla|\nabla\phi|^2) = -\nabla(p + \rho g z).$$

Dividing by ρ, noting that density ρ is constant in the ideal flow and rearranging terms, we get

$$\nabla(-\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2) = -\nabla(\frac{p}{\rho} + gz).$$

Or,

$$\nabla(-\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz) = 0,$$

which implies

$$-\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = C(t), \tag{22}$$

where $C(t)$ is the integration constant which can be function of time only. The above equation is called the **Euler's integral** or as the **Bernoulli's equation** for the ideal flow. Without loss of generality, one can set $C(t)$ to be zero, by appropriately re-defining ϕ:

$$\phi => \phi + \int C(t)\ dt$$

The Euler's integral is therefore given in the familiar form as

$$-\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = 0. \tag{23}$$

Having solved the Laplace equation $\nabla^2\phi = 0$ for ϕ, one can determine the velocity by differentiation of ϕ (as $\vec{u} = -\nabla\phi$); and one can use the above Euler's integral to determine the other unknown, namely the pressure p. One can thus solve an ideal-flow problem.

To solve the Laplace equation, we have to specify appropriate boundary conditions. We shall derive boundary conditions for a general wave-motion problem in the next chapter.