8. REFRACTION & SHOALING OF WAVES APPROACHING LAND

Change in the direction of wave propagation is referred to as wave refraction. Variations in water depth will lead to changes in the wavelength and to refraction. To obtain relations governing refraction, let us first generalize the expressions obtained in the earlier section for wave-number and frequency evolutions in space and time to wave propagation along any direction on the horizontal xy plane. By convention, in the case of waves approaching a shore, the coordinates used are locally rectangular Cartesian, with x axis pointing towards the land, y axis parallel to the land and z axis against gravity (see figure below). Lines indicating the direction of wave propagation (normal to wave crests) are referred to as orthogonals or rays. The angle between the direction of wave propagation (ray) and x axis, denoted by α, is called the angle of incidence.

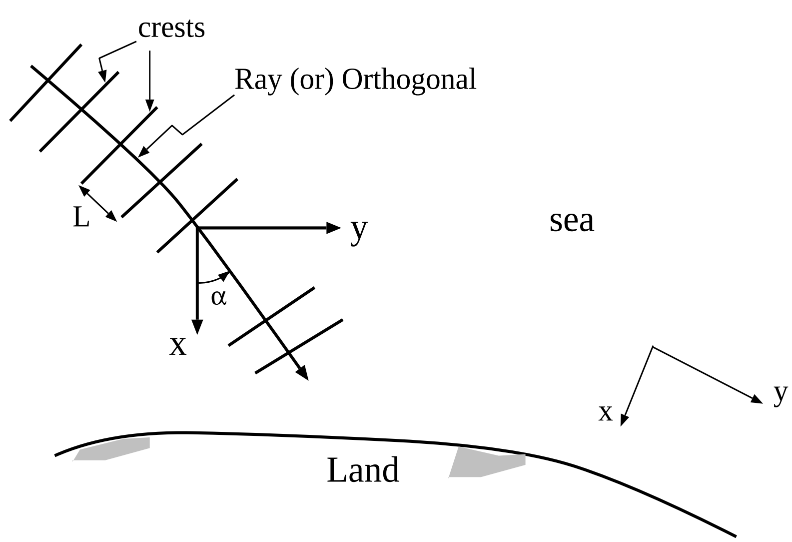

Fig 8-1. Wave refraction: notations and coordinate system

In order to imply the direction of wave propagation also, the wave number is now defined as vector with components along x and y directions; ie.

$$\vec{k} = k_x \hat{\imath} + k_y \hat{\jmath}$$

where $\hat{\imath}$ and $\hat{\jmath}$ denote unit vectors along x and y directions, respectively. The magnitude of $\vec{k}$ denoted simply as k (which is equal to $2\pi/L$) is

$$|\vec{k}| \equiv k = \sqrt{k_x^2 + k_y^2}$$

In terms of k and α, the components of $\vec{k}$ can be written as

$$k_x = k\ \cos\alpha, \quad k_y = k\ \sin\alpha$$

and α in terms of k_x and k_y is given by

$$\alpha = \tan^{-1}\frac{k_y}{k_x}$$

The phase function of waves approaching a shore can now be written as

$$\Theta = \vec{k}\cdot\vec{X} - \sigma t = k_x x + k_y y - \sigma t$$

where $\vec{X} \equiv x\hat{\imath} + y\hat{\jmath}$ and σ denotes the wave radian frequency. Assuming slow variation of $\vec{k}$ and σ with respect to space and time (review earlier section), we can write $\vec{k}$ and σ in terms of Θ as

$$\vec{k} = \nabla_2\Theta; \quad \sigma = -\frac{\partial\Theta}{\partial t}$$

where

$$\nabla_2 \equiv \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}$$

denotes the surface gradient operator. Differentiating $\vec{k}$ with respect to time, σ with respect to space and adding, we obtain

$$\frac{\partial\vec{k}}{\partial t} + \nabla_2\sigma = \frac{\partial}{\partial t}\nabla_2\Theta - \frac{\partial}{\partial t}\nabla_2\Theta = 0$$

Thus we have the following equation governing slow variation of wave number $\vec{k}$ and frequency σ:

$$\frac{\partial\vec{k}}{\partial t} + \nabla_2\sigma = 0$$

Steady-State Waves: In the case of steady-state waves, (ie. $\vec{k}$ and σ independent of time), the above equation simplifies to

$$\nabla_2\sigma = 0$$

which means that σ is independent of (x,y) location also. In other words, in the case of steady-state waves, σ is simply a constant. Note however, in the case of steady-state waves, wave number is independent of time only and can vary with location on the horizontal xy plane.

Refraction of Steady-State Waves on an Ideal Coast: We shall now derive a simple relation for the refraction of waves which is valid under the following assumptions and cases:

- slowly varying waves
- steady state

• ideal coastline

The assumption of slow variation of waves allows us to define wavenumber and frequency as simple derivatives of the phase function. Steady-state implies that frequency is constant and the wave number is independent of time. An ideal coastline refers to shore that is straight and infinitely long with bottom contours (h=constant lines) also being straight and parallel to the shore (see Fig. 8-2). In the case of waves approaching an ideal coast, wave properties are independent of y (ie. properties do not vary along the coast).

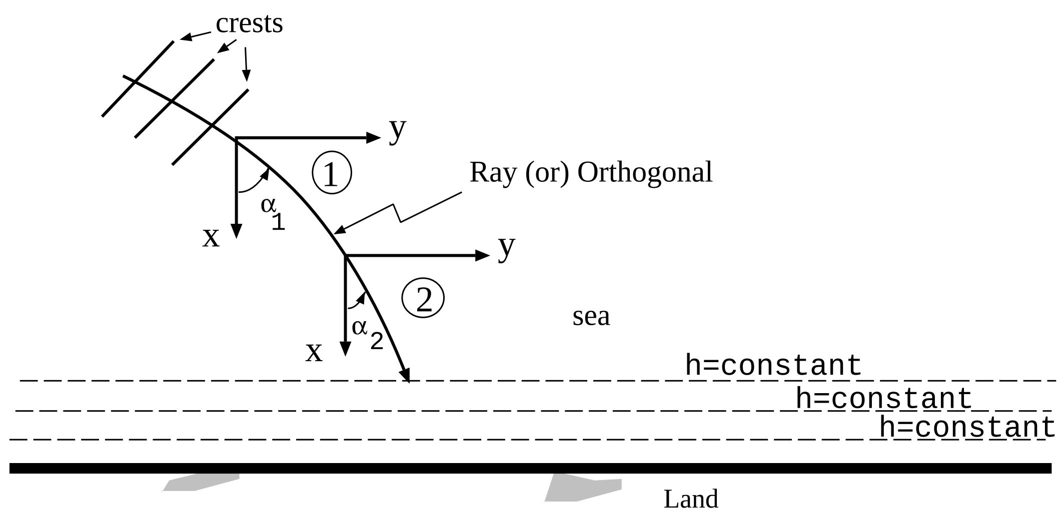

Fig 8-2. Refraction of waves approaching an ideal shore.

Under the assumption of slow variation of waves, we have expressed the wavenumber vector $\vec{k}$ as

$$\vec{k} = \nabla_2 \Theta$$

ie, as a gradient of a scalar. Therefore, curl of $\vec{k}$ is zero; ie.

$$\nabla_2 \times \nabla_2 \Theta = \nabla_2 \times \vec{k} = \frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y} = 0$$

In the case of ideal shore, as things do not vary along the y direction, the above equation reduces to

$$\frac{\partial k_y}{\partial x} = 0$$

which imples k_y can only be a function of y and time t. It cannot be a function of y (as the shoreline is ideal) nor a function of time (because of steady-state). Therefore, k_y (which is equal to k sinα) must be a constant:

$$k_y = k \ \sin\alpha = \ \text{constant}$$

In the case of steady-state waves, σ is also a constant. Dividing the above relation by σ and noting that $\sigma/k \equiv C_p$, we get

$$\frac{k\ \sin\alpha}{\sigma} = \text{constant} \rightarrow \frac{\sin\alpha}{C_p} = \text{constant}$$

which is the familiar Snell's Law of refraction. For example, as shown Fig. 8-2, at two locations indicated by (1) and (2), the angles of incidence are related as

$$\frac{\sin\alpha_1}{C_{p1}} = \frac{\sin\alpha_2}{C_{p2}}$$

Shoaling: Change in Wave Height Next, let us examine the change in height that waves undergo as they approach land. In order to obtain a simple relation, we shall make the following assumptions:

- slowly varying waves
- steady state
- ideal coastline
- no wave breaking

As the coastline is ideal, the rays are geometrically same. The two rays shown in Fig. 8-3 are therefore shifted throughout in the y direction by a distance l.

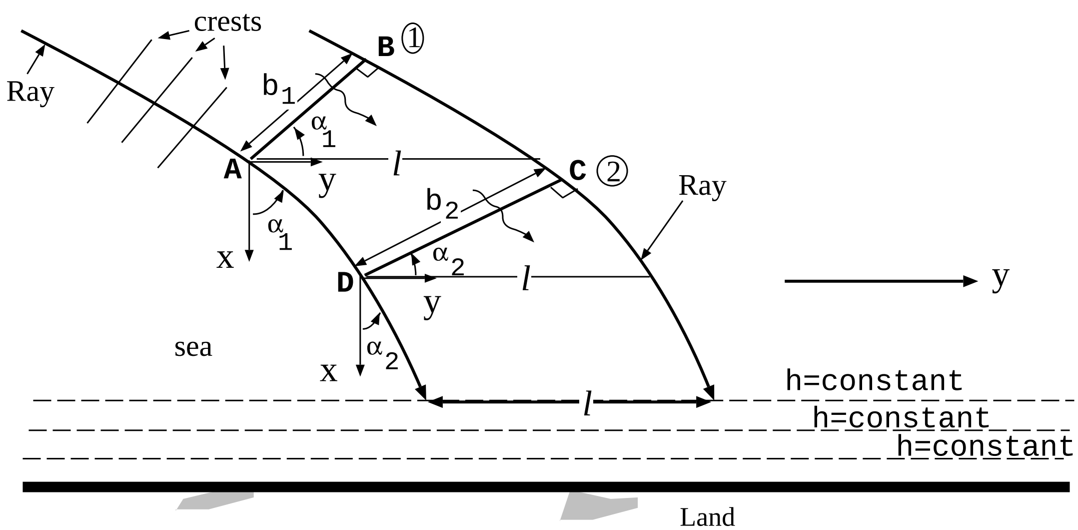

Fig 8-3. Wave Shoaling on an Ideal Shore

Now consider a control region between the rays and locations (1) and (2), designated as ABCD. The average amount of energy entering the control region across AB per unit time must be equal to that leaving the control region CD per unit time, as the waves are in steady state, no wave breaking occurs and energy is conserved. In other words,

$$\bar{\mathcal{F}}_1 b_1 = \bar{\mathcal{F}}_2 b_2$$

where b_1 and b_2 denote the crest width, between the rays, at locations (1) and (2) and $\mathcal{F}$ the average flux of energy per unit crest width. Recall, from earlier section that

$$\mathcal{F} = \bar{E} C_g$$

where $\bar{E} = 1/8\ \rho g H^2$ (average wave energy density) and C_g denotes the group speed. Therefore, flux of energy across the control region governed by

$$\bar{\mathcal{F}}_1 b_1 = \bar{\mathcal{F}}_2 b_2$$

becomes

$$\frac{1}{8}\rho g H_1^2 C_{g1} b_1 = \frac{1}{8}\rho g H_2^2 C_{g2} b_2 \ \rightarrow\ H_1^2 C_{g1} b_1 = H_2^2 C_{g2} b_2$$

In the case of a shoreline with ideal bottom topography, the crestwidths between the rays are given by (see Fig. 8-3)

$$b_1 = l\ \cos\alpha_1, \quad b_2 = l\ \cos\alpha_2.$$

where l is the y-distance of separation between the rays. Thus, we obtain the following relations governing change of wave height:

$$H_1^2 C_{g1} b_1 = H_2^2 C_{g2} b_2 \ \rightarrow H_1^2 C_{g1} l\ \cos\alpha_1 = H_2^2 C_{g2} l\ \cos\alpha_2 \ \rightarrow H_1^2 C_{g1}\ \cos\alpha_1 = H_2^2 C_{g2}\ \cos\alpha_2$$

Or,

$$H_2 = H_1 \sqrt{\frac{C_{g1}}{C_{g2}}} \sqrt{\frac{\mathrm{Cos}\alpha_1}{\mathrm{Cos}\alpha_2}}$$

To sum, in the case steady-state, slowly-varying waves incident on an ideal beach (without breaking), wave refraction and shoaling are governed by the following relations:

$$\frac{\sin\alpha_1}{C_{p1}} = \frac{\sin\alpha_2}{C_{p2}} \quad \text{(Snells Law)} \tag{144}$$

and

$$H_2 = H_1 \sqrt{\frac{C_{g1}}{C_{g2}}} \sqrt{\frac{\mathrm{Cos}\alpha_1}{\mathrm{Cos}\alpha_2}} \tag{145}$$