EOC4422 OCEAN WAVE MECHANICS FALL 2004

3. Periodic Solutions to Linearized Free-Surface Flow Problem

Having formulated the equations governing a general wave motion problem, let us now seek some fundamental, two-dimensional, periodic solutions for "linear" free-surface flow in water of constant depth. These solutions would enable us understand some fascinating properties of surface waves as observed in the ocean.

The problem to be solved in this chapter, which corresponds to two-dimensional wave motion in water of constant depth h, is illustrated in Fig. 3-1 below. We assume waves to be periodic in the x direction with wave length L (or wave number $k \equiv 2\pi/L$) and time harmonic with period T (or radian frequency $\sigma \equiv 2\pi/T$).

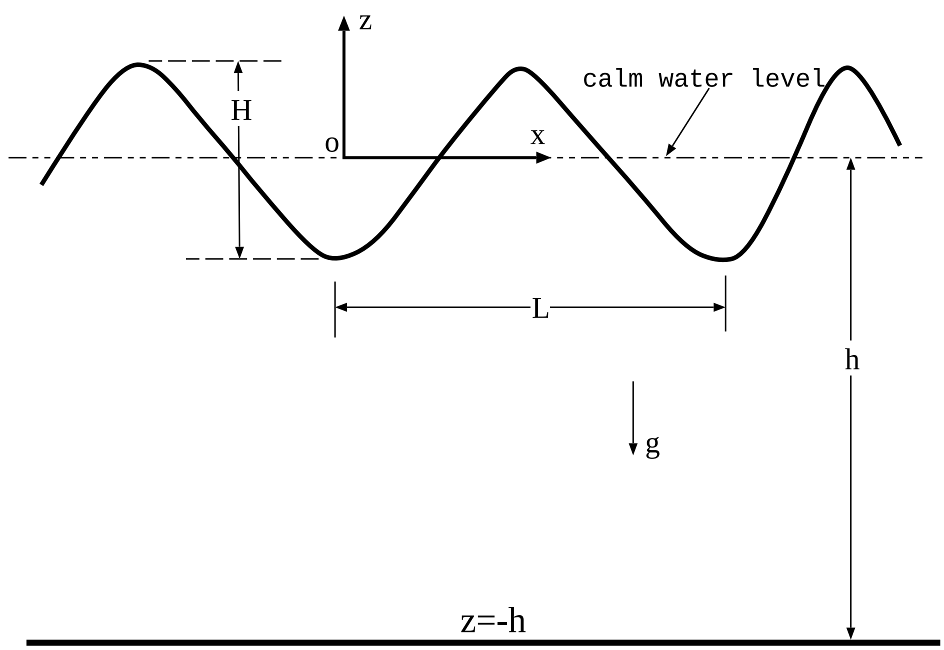

Fig 3-1. Two-dimensional periodic free-surface flow.

Governing Equations

The governing equations for the two-dimensional linear wave motion problem are as follows:

i. Laplace Equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{47}$$

ii. Linearized Euler's Integral

$$p = \rho(\frac{\partial\phi}{\partial t} - gz), \tag{48}$$

iii. Bottom No-Flux Condition

$$\frac{\partial\phi}{\partial z} = 0 \text{ on } z = -h, \tag{49}$$

iv. Linearized Free-Surface Kinematic Condition

$$\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial z} = 0 \text{ on } z = 0, \tag{50}$$

v. Linearized Free-Surface Dynamic Condition (w/o surface tension)

$$-\frac{\partial\phi}{\partial t} + g\eta = 0 \text{ on } z = 0, \tag{51}$$

v. Linearized Free-Surface Combined Condition

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \text{ on } z = 0, \tag{52}$$

vi. Periodicity in x.

$$\phi(x,z,t) = \phi(x+L,zt);\ \eta(x,t) = \eta(x+L,t), \tag{53}$$

vii. Time Harmonic Condition

$$\phi(x,z,t) = \phi(x,zt+T);\ \eta(x,t) = \eta(x,t+T). \tag{54}$$

<u>Separation of Variables</u>

The Laplace equation is separable with respect to rectangular coordinate system. We shall therefore seek solution of the form

$$\phi(x,z,t) \equiv \mathcal{X}(x)\mathcal{Z}(z)\mathcal{T}(t). \tag{55}$$

Substituting the Laplace equation and dividing by $\mathcal{X}\mathcal{Z}\mathcal{T}$, one can obtain

$$\frac{1}{\mathcal{X}}\frac{d^2\mathcal{X}}{dx^2} + \frac{1}{\mathcal{Z}}\frac{d^2\mathcal{Z}}{dz^2} = 0. \tag{56}$$

Since $\mathcal{X}$ is function of x only and $\mathcal{Z}$ of z only, with x and z being independent variables, we have

$$\frac{1}{\mathcal{X}}\frac{d^2\mathcal{X}}{dx^2} = -k^2 \tag{57}$$

$$\frac{1}{\mathcal{Z}}\frac{d^2\mathcal{Z}}{dz^2} = +k^2, \tag{58}$$

where the separation constant k can be (i) zero, (ii) imaginary, or (iii) real. Lets explore the consequence of all these three possibilities. When $k = 0$, we have

$$\mathcal{X} = Ax + B \tag{59}$$

$$\mathcal{Z} = Cz + D, \tag{60}$$

where A, B, C, and D are integration constants. As per above Eqn.(59), the solution of ϕ would be linear in x (as well as in z) but not periodic in x as stipulated in Eqn.(53). The separation constant can not be zero for the present problem.

In the case when k is an imaginary number, the differential equations for $\mathcal{X}$ and $\mathcal{Z}$ become,

$$\frac{1}{\mathcal{X}}\frac{d^2\mathcal{X}}{dx^2} = +|k|^2 \tag{61}$$

$$\frac{1}{\mathcal{Z}}\frac{d^2\mathcal{Z}}{dz^2} = -|k|^2, \tag{62}$$

where $|k|$ is the modulus of k. The solution for the above differential equations are given by

$$\mathcal{X} = Ae^{|k|x} + Be^{-|k|x} \tag{63}$$

$$\mathcal{Z} = C\ \sin(|k|z) + D\ \cos(|k|z). \tag{64}$$

This solution is periodic in z but not in x as required by Eqn.(53). Therefore, imaginary k is also admissible for the present problem.

Finally, for k being real the solution of Eqn.(57) is given by

$$\mathcal{X} = A\ \sin(kx) + B\ \cos(kx) \tag{65}$$

$$\mathcal{Z} = Ce^{kz} + De^{-kz}. \tag{66}$$

As per the separation (Eqn. 55), ϕ is then

$$\phi = \{A\ \sin(kx) + B\ \cos(kx)\}(Ce^{kz} + De^{-kz})\mathcal{T}(t). \tag{67}$$

We now proceed to determine the integration constants A, B, C, and D as well as $\mathcal{T}(t)$ in Eqn.(67). Using the bottom no-flux boundary condition Eqn.(49), we obtain

$$\begin{aligned}\frac{\partial\phi}{\partial z} &= 0 \text{ on } z = -h \\ &=> \{A\ \sin(kx) + B\ \cos(kx)\}(Cke^{-kh} - Dke^{kh})\mathcal{T}(t) = 0 \\ &=> C = De^{2kh}\end{aligned} \tag{68}$$

With the above relation between C and D, ϕ can be written as

$$\begin{aligned}\phi &= \{A\ \sin(kx) + B\ \cos(kx)\}(Ce^{kz} + De^{-kz})\mathcal{T}(t) \\ &= \{A\ \sin(kx) + B\ \cos(kx)\}(De^{2kh}e^{kz} + De^{-kz})\mathcal{T}(t) \\ &= \{A\ \sin(kx) + B\ \cos(kx)\}De^{kh}(e^{k(h+z)} + e^{-k(h+z)})\mathcal{T}(t) \\ &= \{A\ \sin(kx) + B\ \cos(kx)\}De^{kh}2\ \text{Cosh}k(h+z)\mathcal{T}(t).\end{aligned} \tag{69}$$

Substituting this expression for ϕ in the free-surface combined condition Eqn.(52), we obtain

$$\begin{aligned}\frac{\partial^2\phi}{\partial t^2} &+ g\frac{\partial\phi}{\partial z} = 0 \text{ on } z = 0 \\ &=> \{A\ \sin(kx) + B\ \cos(kx)\}De^{kh}2[\text{Cosh}kh\ \frac{d^2\mathcal{T}}{dt^2} + gk\ \text{Sinh}(kh)\ \mathcal{T}] = 0 \\ &=> \frac{d^2\mathcal{T}}{dt^2} = -gk\ \text{Tanh}kh\ \mathcal{T}.\end{aligned} \tag{70}$$

Solution of $\frac{d^2\mathcal{T}}{dt^2} = -gk\ \mathrm{Tanh}kh\ \mathcal{T}$ is given by

$$\mathcal{T} = E\ \sin(\sigma t) + F\ \cos(\sigma t), \tag{71}$$

where E and F are integration constants, and

$$\sigma^2 = gk\ \mathrm{Tanh}kh. \tag{72}$$

Because of periodicity in time the above σ is nothing but the radian frequency of the wave motion! The above equation which relates wave frequency σ and wave number k is known as the dispersion relation.

The solution for ϕ

$$\phi = \{A\ \sin(kx) + B\ \cos(kx)\}De^{kh}2\ \mathrm{Cosh}k(h+z)\{E\ \sin(\sigma t) + F\ \cos(\sigma t)\} \tag{73}$$

has four terms in it, each satisfying the governing equations. For convenience, combining the constants, the expression for ϕ can be written as

$$\begin{aligned}\phi \ = \ & A_1\ \mathrm{Cosh}k(h+z)\ \sin(kx)\ \sin(\sigma t) \\ & +A_2\ \mathrm{Cosh}k(h+z)\ \sin(kx)\ \cos(\sigma t) \\ & +A_3\ \mathrm{Cosh}k(h+z)\ \cos(kx)\ \sin(\sigma t) \\ & +A_4\ \mathrm{Cosh}k(h+z)\ \cos(kx)\ \cos(\sigma t)\end{aligned} \tag{74}$$

For convenience, we shall pursue further analysis for only one of the four terms in the above equation. Analysis for other terms would be identical, and, because of linearity, any linear combination of these solutions will also be another solution to the present linear, homogeneous, free-surface flow problem.

Standing Wave Solutions

Let

$$\phi = A_1\ \mathrm{Cosh}k(h+z)\ \sin(kx)\ \sin(\sigma t) \tag{75}$$

Using the free-surface dynamic condition (Eqn. 51), we can determine the free-surface displacement corresponding to the above potential:

$$\begin{aligned}\eta &= \frac{1}{g}\frac{\partial \phi}{\partial t} \text{ on } z = 0 \\ &= \frac{1}{g}A_1\sigma\ \mathrm{Cosh}kh\ \sin(kx)\ \cos(\sigma t)\end{aligned} \tag{76}$$

Setting the above, in terms of wave height H, as

$$\eta = \frac{H}{2}\ \sin(kx)\ \cos(\sigma t), \tag{77}$$

we find A_1 to be

$$A_1 = \frac{H}{2}\frac{g}{\sigma}\frac{1}{\mathrm{Cosh}kh}\frac{H}{2}\ \sin(kx)\ \cos(\sigma t). \tag{78}$$

The expression for velocity potential is therefore

$$\phi = \frac{H}{2}\frac{g}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\ \sin(kx)\ \sin(\sigma t). \tag{79}$$

One can easily check that above solution, lets call it **[S1]**,

$$\begin{aligned} \eta &= \frac{H}{2}\ \sin(kx)\ \cos(\sigma t), && \textbf{[S1]} \\ \phi &= \frac{H}{2}\frac{g}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\ \sin(kx)\ \sin(\sigma t) \end{aligned}$$

represents a standing wave. The nodes of the free-surface correspond to $kx = n\pi$ where n is an integer, and the anti-nodes correspond to $kx = n\pi/2$. Corresponding to remaining terms of Eqn. (74) one can likewise obtain standing-wave solutions. Instead of repeating the same steps, we simply list these standing wave solutions below and call these solutions as **[S2], [S3]**, and **[S4]** for easier reference later.

$$\begin{aligned} \eta &= \frac{H}{2}\ \sin(kx)\ \sin(\sigma t), && \textbf{[S2]} \\ \phi &= -\frac{H}{2}\frac{g}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\ \sin(kx)\ \cos(\sigma t), \end{aligned}$$

$$\begin{aligned} \eta &= \frac{H}{2}\ \cos(kx)\ \cos(\sigma t), && \textbf{[S3]} \\ \phi &= \frac{H}{2}\frac{g}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\ \cos(kx)\ \sin(\sigma t) \end{aligned}$$

$$\begin{aligned} \eta &= \frac{H}{2}\ \cos(kx)\ \sin(\sigma t), && \textbf{[S4]} \\ \phi &= -\frac{H}{2}\frac{g}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\ \cos(kx)\ \cos(\sigma t) \end{aligned}$$

As per *superposition principle* valid for linear problem, any linear combination of these solutions,

$$(B_1 \times [\textbf{S1}]) + (B_2 \times [\textbf{S2}]) + (B_3 \times [\textbf{S3}]) + (B_4 \times [\textbf{S4}])$$

where B_1, B_2, B_3, and B_4 are arbitrary constants, will be another solution to the linear, homogeneous free-surface flow problem. We shall consider such summations and construct progressive wave solutions in the next section.

Progressive Wave Solutions

By summing the standing wave solutions **[S2]** and **[S3]** and using the trigonometric identities $\sin(kx-\sigma t) \equiv \sin(kx)\ \cos(\sigma t) - \cos(kx)\ \sin(\sigma t)$ and $\cos(kx-\sigma t) \equiv \cos(kx)\ \cos(\sigma t) + \cos(kx)\ \sin(\sigma t)$, we obtain following solution:

$$\begin{aligned} \eta &= \frac{H}{2}\ \cos(kx-\sigma t), && \textbf{[P1]} \\ \phi &= -\frac{H}{2}\frac{g}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\ \sin(kx-\sigma t) \end{aligned}$$

which, with k and σ being positive, represent a wave traveling along the positive x direction. We have denoted this *progressive* (meaning traveling) wave solution as **[P1]**.

Similarly, by considering various linear combinations of standing wave solutions **[S1]**, **[S2]**, **[S3]** and **[S4]**, one can obtain the following progressive wave solutions denotes as **[P2]**, **[P3]** and **[P4]**:

$$\begin{aligned} \eta &= \frac{H}{2}\cos(kx+\sigma t), && \textbf{[P2]} \\ \phi &= \frac{H}{2}\frac{g}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\sin(kx+\sigma t) \end{aligned}$$

$$\begin{aligned} \eta &= \frac{H}{2}\sin(kx-\sigma t), && \textbf{[P3]} \\ \phi &= \frac{H}{2}\frac{g}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\cos(kx-\sigma t) \end{aligned}$$

$$\begin{aligned} \eta &= \frac{H}{2}\sin(kx+\sigma t), && \textbf{[P4]} \\ \phi &= -\frac{H}{2}\frac{g}{\sigma}\frac{\text{Cosh}k(h+z)}{\text{Cosh}kh}\cos(kx+\sigma t). \end{aligned}$$

Note that wave with the phase function $(kx+\sigma t)$ represent a wave traveling in the negative x direction and with the phase function $(kx-\sigma t)$ in the positive x direction. It is a simple exercise to show that one can construct standing wave solutions also by linearly combining the above progressive wave solutions. For example, two progressive waves of equal amplitude traveling in the opposite directions would set up a standing wave of twice the amplitude! You may want to verify this as a home work.

Phase Speed or Celerity

By *phase* speed, denoted as C_p, we mean the speed of the frame of reference in which progressive wave would appear stationary. For example, consider a progressive wave with phase function (usually denoted as θ) of the form $(kx-\sigma t)$. For the phase to be constant,

$$\begin{aligned} d\theta &= d(kx-\sigma t) \\ &= kdx-\sigma dt = 0 \text{ since } k \text{ and } \sigma \text{ are constants.} \\ => \frac{dx}{dt} &= \frac{\sigma}{k} \equiv C_p \end{aligned}$$

In view of the dispersion relation, Eqn.(72), the phase speed can be expressed as,

$$C_p = \sqrt{\frac{g}{k}\,\text{Tanh}kh} \tag{80}$$

We thus observe that speed of water waves depends not only on the wave number but also on the water depth. We shall discuss this and other properties further in the following chapters.