Lecture #15

Surface Waves

In this chapter, we shall study gravity waves on the surface of the ocean which is an importans means of transport of energy in the ocean and transfer of momentum and energy between the atmosphere and the ocean. Waves of numerous lengths, ranging from a millimeter to hundreds of kilometer, exist on the surface of the ocean. Depending on the length scale and velocity (or frequency) scale, different effects become important of insignificant. For example, on tiny waves of lengths of the order of centimeter or less (known as capillary waves), the effect of surface tension is quite significant; whereas on waves of lengths of the order of meter or more, the effect of surface tension is insignificant compared to the gravitational effect. Similarly on very long waves, spanning the entire ocean, the effect of Coriolis acceleration is significan. On the other hand, on waves of lengths in the range of O(1[m]) to O(1000 [m]) and velocities in the range of O(0.1 [m/s]) to O(1 [m/s]) or more, the effect of Coriolis acceleration on the wave motions is insignificant. In this chapter, we shall consider surface waves in which Coriolis acceleration is insignificant and the dynamics is governed by inertia and gravity. We shall call these surface gravity waves and study their fascinating properties. The theory of waves has always provided great impetus to applied mathematics. In this chapter, we shall try to derive all the relations govering the gravity waves starting from fundamental principles and using techniques of applied mathematics, of course without losing the sight on the objective of the course! Those interested to study this subject further, will find very many classical texts written on the subject.

As stated above, we shall formulate the problem governing surface waves by assuming that the earth-fixed coordinate system is inertial. As our focus now is on understanding only some basic kinematic and kinetic properties of the waves, we shall make additional assumptions such as that the effect of viscosity is negligible and that the flow is irrotational. This assumption, known as the ideal-fluid assumption, applies well for waves with amplitude smaller than the length. The effect of viscosity is confined only within a thin boundary layer on the sea surface and sea bottom. The ideal flow solution is applicable in the bulk of the waters exterior to the boundary layer. The equations governing the wave motion are therefore,

$$\nabla \cdot \vec{u} = 0 \text{ (the equation of continuity)} \tag{1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\nabla \cdot \vec{u})\vec{u} = -\frac{1}{\rho}\nabla(p + \rho g z) \text{ (the Eulers equation)} \tag{2}$$

As viscosity is the primary means of the generation of vorticity, we can further assume that the flow is irrotational; ie., $\nabla \times \vec{u} = 0$. One formally show that the velocity field of an irrotational flow can be represented as a gradient of a scalar field known as the velocity potential;

$$\vec{u} = \nabla \phi$$

where ϕ denotes the velocity potential. Substituting $\nabla\phi$ for $\vec{u}$, and noting that $(\nabla \cdot \vec{u})\vec{u} \equiv \frac{1}{2}\nabla|\vec{u}|^2 - \vec{u} \times (\nabla \times \vec{u}) = \frac{1}{2}\nabla|\vec{u}|^2$ as $\nabla \times \vec{u} = 0$ by assumption, we can obtain the following equations from the equation of continuity and the Eulers equation:

$$\nabla^2\phi = 0 \ \ \text{(known as the Laplace equation)} \tag{3}$$

and

$$\nabla\{\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + p + \rho g z\} = 0 \tag{4}$$

or,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + p + \rho g z = 0, \ \text{(known as the Eulers Integral)} \tag{5}$$

Note that in obtaining the Eulers integral, we have set the time constant to be zero. One can formally show that this does not cause any loss of generality, by redefining ϕ; but let us not delve into such fine points now.

The recipe for the solution of an ideal flow is as follows. First solve the Laplace equation subject to appropriate boundary conditions (to be discussed later for the wave problem). Having solved for ϕ, one determines velocity by differentiation $\vec{u} = \nabla\phi$ and the pressure using the Eulers integral. The solution procedure for an ideal flow is thus very straightforward compared to that for the Navier-Stokes equation or even the Eulers equation.

Boundary Conditions for Wave Motion

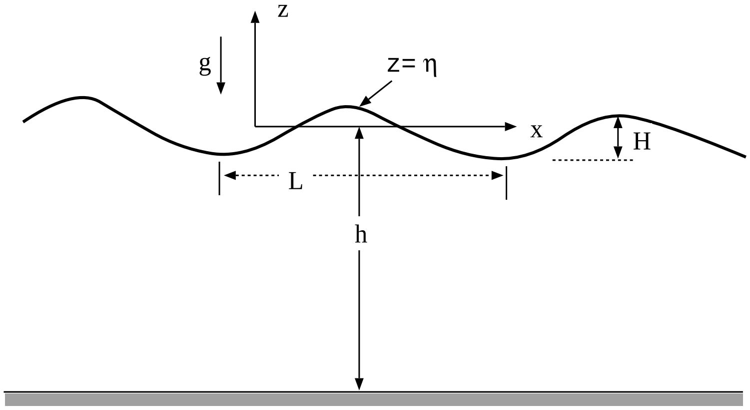

Fig 15-1. Coordinate system and notations used for problem formulation

Let us consider a two-dimensional wave motion as depicted in the figure above. Note that the coordinate system oxz has its origin on the mean (or calm) surface with the z axis pointing against the gravity. Let h (=constant) denote the water depth, H the wave height, L the wave length and $z = \eta = \eta(x,t)$ the free-surface elevation. The boundary conditions for the wave motion are as follows. On the bottom boundary, which is assumed to be stationary, horizontal and impermeabel, the flux of water across the surface is zero; in other words

$$w = \frac{\partial \phi}{\partial z} = 0 \text{ on z = -h}$$

The sea surface, which is free to deform and hence also referred to as a free surface, is a material surface. In other words, particles on the surface will always remain on the surface. Mathematically, this kinematic condition implies

$$\frac{D[z - \eta(x,t)]}{Dt} = 0$$

Expanding, we obtain

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial z}, \text{ on z=}\eta$$

which is nonlinear as both ϕ and η are unknowns and as the condition need to be satisfied on the unknown surface $z = \eta$.

free-surface conditions is presented next.

Linearized Free-Surface Boundary Conditions

We shall assume that wave motion is "small". In particular, we shall assume that wave deformation and fluid velocity to be so small that we can ignore the products of η, ϕ, and their derivatives. Without such product terms, the free-surface kinematic and dynamic conditions

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = 0 \text{ on } z = \eta \tag{6}$$

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \text{ on } z = \eta. \tag{7}$$

The quantities $\frac{\partial \phi}{\partial z}$ and $\frac{\partial \phi}{\partial t}$ appearing in these equations have to be still evaluated on unknown $z = \eta$. One can surmount this difficulty by Taylor expanding these terms about $z = 0$ (calm surface) and neglecting products of ϕ and η terms as done above, and get

$$\begin{aligned} \frac{\partial \phi}{\partial z}(x,t,z=\eta,t) &= \frac{\partial \phi}{\partial z}(x,t,z=0,t) + \frac{\eta}{2!}\frac{\partial^2 \phi}{\partial z^2}(x,t,z=0,t) + ... &(8)\\ &\approx \frac{\partial \phi}{\partial z}(x,t,z=0,t) &(9)\end{aligned}$$

$$\begin{aligned} \frac{\partial \phi}{\partial t}(x,t,z=\eta,t) &= \frac{\partial \phi}{\partial t}(x,t,z=0,t) + \frac{\eta}{2!}\frac{\partial^2 \phi}{\partial z \partial t}(x,t,z=0,t) + ... &(10)\\ &\approx \frac{\partial \phi}{\partial t}(x,t,z=0,t). &(11)\end{aligned}$$

The free-surface conditions to the leading order thus become

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = 0 \text{ and } \frac{\partial \phi}{\partial t} + g\eta = 0 \text{ on } z = 0. \tag{12}$$

These are the linearized free-surface conditions, which need to be satisfied merely on the calm surface $z = 0$. Also note one can eliminate η by combining time-derivative of these equations, and get

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \text{ on } z = 0. \tag{13}$$

This equation for ϕ is known as the *combined linear free-surface condition.* It is important to have in mind that solutions obtained based on these linear free-surface conditions will be valid only for *small-amplitude waves.* By "small-amplitude waves" we mean that wave-height/wave-length ratio is $<< 1$.