Lecture #14

Vorticity Equation

In this chapter we will derive exact and approximate vorticity equations, using which one can explain the phenomenon of westward (eastward) intensification of currents in the northern (southern) hemisphere. You will notice in the figures of surface currents given in the text and other reference books, that currents are stronger as they flow along western boundary of an ocean in the northern hemisphere (eg., Gulf stream and Kuroshio current); similar phenomenon occurs in the southern hemisphere, but along the eastern boundary of an ocean. There are models which can predict the occurrence of westward or eastward intensification. Here, we use vorticity equation and the conservation potential vorticity (meaning of which will become clear later) to qualitatively predict the occurrence of westward intensification.

To derive the vorticity equation, we begin with the vector form of the Navier-Stokes equation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} + 2\vec{\Omega}\times\vec{u} = -\frac{1}{\rho}\nabla p - \nabla gz + A\nabla^2\vec{u}$$

Using a vector identity

$$(\vec{u}\cdot\nabla)\vec{u} = (\nabla\times\vec{u})\times\vec{u} - \frac{1}{2}\nabla|\vec{u}|^2$$

the momentum equation can be written as

$$\frac{\partial \vec{u}}{\partial t} + (\nabla\times\vec{u})\times\vec{u} - \frac{1}{2}\nabla|\vec{u}|^2 + 2\vec{\Omega}\times\vec{u} = -\frac{1}{\rho}\nabla p - \nabla gz + A\nabla^2\vec{u}$$

The term $\nabla\times\vec{u}$ denoted as $\vec{\zeta}$ is called the relative vorticity, because the term represents the vorticity of the flow as observed from a rotating frame of reference. The term $2\vec{\Omega}$, where $\vec{\Omega}$ is the angular velocity of the earth is also called the planetary vorticity. Recall from elementary kinematics of fluids that vorticity is twice the angular velocity. The sum $\vec{\zeta} + 2\vec{\Omega}$ denoted as $\vec{\zeta}_a$ is called the absolute vorticity. With respect to a fixed inertial frame of reference the vorticity of the flow will be $\vec{\zeta}_a$, and hence the name for $\vec{\zeta}_a$.

With above definitions, the momentum equation can be written as

$$\frac{\partial \vec{u}}{\partial t} + (\vec{\zeta} + 2\vec{\Omega})\times\vec{u} - \frac{1}{2}\nabla|\vec{u}|^2 = -\frac{1}{\rho}\nabla p - \nabla gz + A\nabla^2\vec{u}$$

$\rightarrow$

$$\frac{\partial \vec{u}}{\partial t} + \vec{\zeta}_a\times\vec{u} - \frac{1}{2}\nabla|\vec{u}|^2 = -\frac{1}{\rho}\nabla p - \nabla gz + A\nabla^2\vec{u}$$

Taking the *curl* of the above equation, we obtain

$$\frac{\partial \vec{\zeta}}{\partial t} + \nabla\times(\vec{\zeta}_a\times\vec{u}) = -\nabla(\frac{1}{\rho})\times\nabla p + A\nabla^2\vec{\zeta}$$

as curl of a gradient is identically zero. Since $2\vec{\Omega}$ is a constant, $\frac{\partial 2\vec{\Omega}}{\partial t} = 0$ and $A\nabla^2 2\vec{\Omega} = 0$. Adding the trivial terms and noting that $\vec{\zeta_a} = \vec{\zeta} + 2\vec{\Omega}$, we obtain

$$\frac{\partial \vec{\zeta_a}}{\partial t} + \nabla \times (\vec{\zeta_a} \times \vec{u}) = -\nabla(\frac{1}{\rho}) \times \nabla p + A\nabla^2 \vec{\zeta_a}$$

By a vector identity, the term $\nabla \times (\vec{\zeta_a} \times \vec{u})$can be expanded into

$$\nabla \times (\vec{\zeta_a} \times \vec{u}) \equiv (\vec{u} \cdot \nabla)\vec{\zeta_a} - (\vec{\zeta_a} \cdot \nabla)\vec{u} + \vec{\zeta_a}\nabla \cdot \vec{u} - \vec{u}\nabla \cdot \vec{\zeta_a}$$

In the above the last two terms are zero, because (i) the fluid is incompressible ($\nabla \cdot \vec{u} = 0$) and (ii) $\nabla \cdot \vec{\zeta_a} = \nabla \cdot (\nabla \times \vec{u} + 2\vec{\Omega})$ which is also zero as divergence of a curl is zero and $\vec{\Omega}$ is a constant. Substituting the expansion for $\nabla \times (\vec{\zeta_a} \times \vec{u})$ in the earlier equation and re-arranging terms, we get

$$\frac{\partial \vec{\zeta_a}}{\partial t} + (\vec{u} \cdot \nabla)\vec{\zeta_a} = (\vec{\zeta_a} \cdot \nabla)\vec{u} - \nabla(\frac{1}{\rho}) \times \nabla p + A\nabla^2 \vec{\zeta_a}$$

$\rightarrow$

$$\frac{D\vec{\zeta_a}}{Dt} = (\vec{\zeta_a} \cdot \nabla)\vec{u} - \nabla(\frac{1}{\rho}) \times \nabla p + A\nabla^2 \vec{\zeta_a}$$

which is called the vorticity equation. The left-hand side of the equation denotes the rate of change of particle vorticity. The could be due to stretch or tilt of fluid elements, as represented by the first term on the right, due to *baroclinic torque* denoted by the second term, and due to diffusion as shown by the last term on the right-hand side. Note that the first term on the right representing stretch/tilt is a three-dimensional effect. In a homogeneous flow, the baroclinic torque will be zero as $\nabla(1/\rho) = 0$. The above vorticity equation is the most general for an incompressible fluid. The equation is however not trivial to solve. In the following section, we make a few assumptions justifiable for oceanographic flows and derive a simple equation for what is known as the *potential vorticity.* The conservation of potential vorticity will be used to explain qualitatively the westward intensification of ocean currents.

Potential Vorticity

Let us assume the following to reduce the vorticity equation to a simple relation valid for fluid elements or columns:

- Fluid is homogenious
- Diffusion is negligible
- $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are negligible; in other words, the vertical velocity is small and horizontal length scales are large.
- Horizontal velocity is independent of depth; in other words, $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$ are small.

With above assumptions, the above vorticity transport equation becomes

$$\frac{D\vec{\zeta}_a}{Dt} - (\vec{\zeta}_a \cdot \nabla)\vec{u} = 0$$

Noting that the components of $2\vec{\Omega}$ are $(0,\ 2\Omega\ \cos\theta,\ 2\Omega\ \sin\theta) = (0,\ 2\Omega\ \cos\theta,\ f)$, and observing above assumptions, we have the following for the z component of the equation:

$$\frac{D}{Dt}(\zeta_z + f) - (\zeta_z + f)\frac{\partial w}{\partial z} = 0$$

Integrating the above equation in the z direction over a vertical material column of water height h we get,

$$\int_h^0 \frac{D}{Dt}(\zeta_z + f)\ dz - \int_h^0 (\zeta_z + f)\frac{\partial w}{\partial z}\ dz = 0$$

By the assumptions, the above equation becomes

$$\frac{D}{Dt}(\zeta_z + f) \int_h^0 dz - (\zeta_z + f) \int_h^0 \frac{\partial w}{\partial z}\ dz = 0$$

$\rightarrow$

$$h.\frac{D}{Dt}(\zeta_z + f) - (\zeta_z + f)\{w(z=-h) - w(z=0)\} = 0$$

$\rightarrow$

$$h.\frac{D}{Dt}(\zeta_z + f) - (\zeta_z + f)\frac{Dh}{Dt} = 0$$

(because it is a material column of fluid, $\frac{Dh}{Dt} = \{w(z=-h) - w(z=0)\}$.) Dividing by h^2,

$$\frac{1}{h}\frac{D}{Dt}(\zeta_z + f) - \frac{1}{h^2}\frac{Dh}{Dt} = 0$$

or,

$$\frac{D}{Dt}[\frac{\zeta_z + f}{h}] = 0$$

In other words, the quantity $\frac{\zeta_z+f}{h}$ of a material column of fluid is conserved (ie., remains constant). Such property of vorticity is true in ideal flow, and hence the quantity is called the *potential vorticity*. Using the potential vorticity equation,

$$\frac{D}{Dt}[\frac{\zeta_z + f}{h}] = 0$$

one can explain many interesting phenomena of rotating flows occurring in nature. For example, consider a rotating column of fluid moving *zonally* (along east-west direction) such that the height of it reduces. In this case, f remains the same, h reduces, and therefore for the conservation of potential vorticity $\frac{\zeta_z+f}{h}$, the vertical component of relative voriticy ζ_z must also reduce; in other words, the rate of rotation will drop as the column moves zonally with decreasing h. Next, consider a *meridional* flow, from south to north, of a column of fluid with height h remaining constant. In the case, h remains the same and f increases as the motion is from south to north. Per conservation of potential vorticity, in this case too, the relative vorticity ζ_z decreases as the column moves northward. In the class, using the potential vorticity equation and considering the supply of vorticity by wind and the generation of vorticity in the lateral boundary layer, we will discuss the phenomenon of *westward (eastward) intensification* of a *gyre* in the northern (southern) hemisphere.
