Lecture #13

Wind-Driven Current: Sverdrup's Equation

Next, let us consider another model, which was developed by Sverdrup, for surface currents. The model is more general that that of Ekman, in that the Sverdrup's model retains the horizontal pressure-gradient terms in the momentum equations, allows variability of the wind stress, allows presence of a lateral boundary and it is formulated in the β plane. However, the Sverdrup's equation are for integrated quantities M_x and M_y and not for the velocity components. Further, as in the case of Ekman's equations, the vertical component of velocity is considered to be negligibly small.

The Sverdrup's equations are derived as follows. Based on the usual order-of-magnitude arguments, and above stated assumptions, the governing equations can be reduced to

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial p}{\partial x} = \rho f v + \frac{d\tau_{xz}}{dz} \tag{2}$$

$$\frac{\partial p}{\partial x} = -\rho f u + \frac{d\tau_{yz}}{dz} \tag{3}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{4}$$

The equation of continuity is considered as $\nabla \cdot \vec{u} = 0$, as the flow is steady. Of course, $\nabla \cdot \vec{u} = 0$, as the flow is incompressible, but the former form is preferred in the present formulation and the reason will become clear soon. Note that the vertical diffusion of horizontal momentum are given as $\frac{d\tau_{xz}}{dz}$ and $\frac{d\tau_{yz}}{dz}$, for the following reason.

$$\tau_{xz} = \rho A_z (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) = \rho A_z \frac{\partial u}{\partial z} \quad \rightarrow \frac{d\tau_{xz}}{dz} = \rho A_z \frac{\partial^2 u}{\partial z^2},$$

and

$$\tau_{yz} = \rho A_z (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) = \rho A_z \frac{\partial v}{\partial z} \quad \rightarrow \frac{d\tau_{yz}}{dz} = \rho A_z \frac{\partial^2 v}{\partial z^2}$$

as w is assumed to be zero and that eddy-viscosity coefficient to be constant.

Integrating Eq. (1) with respect to z from a depth $z = -H$ (a constant) to sea surface $z = 0$, where H is suffiently large so that the velocity of the wind-driven current is practically zero, we obtain

$$\int_{-H}^{0} \frac{\partial \rho u}{\partial x}\, dz + \int_{-H}^{0} \frac{\partial \rho v}{\partial y}\, dz = 0 \ \rightarrow \frac{\partial}{\partial x}\int_{-H}^{0} \rho u\, dz + \frac{\partial}{\partial v}\int_{-H}^{0} \rho v\, dz = 0 \ \rightarrow \frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0 \tag{5}$$

Note, by definition, the rates of mass transport along x and y per unit width of a control surface are given by

$$M_x \equiv \int_{H}^{0} \rho u\, dz \text{ and} M_y \equiv \int_{H}^{0} \rho v\, dz$$

Next similarly integrating the x and y components of the momentum equation, we obtain:

$$\int_H^0 \frac{\partial p}{\partial x}\, dz = \int_H^0 \rho f v\, dz + \int_H^0 \frac{d\tau_{xz}}{dz}\, dz \;\rightarrow \frac{\partial}{\partial x}\int_H^0 p\, dz = fM_y + \tau_{x\eta} \tag{6}$$

$$\int_H^0 \frac{\partial p}{\partial y}\, dz = -\int_H^0 \rho f u\, dz + \int_H^0 \frac{d\tau_{yz}}{dz}\, dz \;\rightarrow \frac{\partial}{\partial y}\int_H^0 p\, dz = -fM_x + \tau_{y\eta} \tag{7}$$

as the stress components $\tau_{xz} = \tau_{yz} = 0$ at $z = -H$ (flow vanishes at such great depth). On the surface $z = 0$, stress beneath is equal to that of the wind above the free surface. In other words, $\tau_{xz} = \tau_{x\eta}$ and $\tau_{yz} = \tau_{y\eta}$ on $z = 0$ where $\tau_{x\eta}$ and $\tau_{y\eta}$ are the components of the wind stress vector: $\vec{\tau}_\eta = \tau_{x\eta}\hat{i} + \tau_{y\eta}\hat{j}$

Differentiating Eq.(6) with respect to y and Eq.(7) with respect to x and subtracting, we can eliminate the pressure terms and obtain

$$-\frac{\partial}{\partial x}(fM_x) + \frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial}{\partial y}(fM_y) - \frac{\partial \tau_{x\eta}}{\partial y} = 0 \tag{8}$$

Rearranging terms and noting that $\frac{\partial f}{\partial y} \equiv \beta$, we can rewrite the above equation as

$$-f(\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y}) - \beta M_y + +(\frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial \tau_{x\eta}}{\partial y}) = 0 \tag{9}$$

Using Eq.(5) and moving terms around, we get

$$\beta M_y = (\frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial \tau_{x\eta}}{\partial y}) \tag{10}$$

which can be re-written as

$$\beta M_y = \mathrm{Curl}_z \vec{\tau}_\eta \tag{11}$$

Given the wind-stress vector $\vec{\tau}_\eta$, one can thus determine M_x and M_y using Eq.(11), for M_y, and then having determined M_y using Eq.(5), for M_x:

$$\begin{aligned} M_y &= \frac{1}{\beta}\,\mathrm{Curl}_z \vec{\tau}_\eta \\ \frac{\partial M_x}{\partial x} &= -\frac{\partial M_y}{\partial y} \end{aligned} \tag{12}$$

Note that the integration of the second equation for M_x will introduce a constant which is to be determined based on a boundary condition for M_x; in other words, the formulation allows for the presence of one lateral boundary (land mass).

Analysis of Sverdrup equation

Next we present the Sverdrup's equations in the expanded form. As

$$\beta = \frac{\partial f}{\partial y} = \frac{1}{R}\frac{d}{d\theta} 2\Omega\,\sin\theta = \frac{2\Omega}{R}\,\cos\theta,$$

$$M_y = \frac{R}{2\Omega\cos\theta}(\frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial \tau_{x\eta}}{\partial y})$$

and

$$\frac{\partial M_x}{\partial x} = -\frac{\partial M_y}{\partial y} = -\frac{\partial}{\partial y}\{\frac{1}{\beta}(\frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial \tau_{x\eta}}{\partial y})\}$$

$$= -\frac{1}{\beta}\{\frac{\partial^2 \tau_{y\eta}}{\partial x \partial y} - \frac{\partial^2 \tau_{x\eta}}{\partial y^2}\} + (\frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial \tau_{x\eta}}{\partial y})(\frac{1}{\beta^2})\frac{\partial \beta}{\partial y}$$

$$= -\frac{1}{\beta}\{\frac{\partial^2 \tau_{y\eta}}{\partial x \partial y} - \frac{\partial^2 \tau_{x\eta}}{\partial y^2}\} + (\frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial \tau_{x\eta}}{\partial y})(\frac{1}{\beta^2})\frac{1}{R}\frac{d}{d\theta}\frac{2\Omega}{R}\ \cos\theta$$

$$= -\frac{1}{\beta}\{\frac{\partial^2 \tau_{y\eta}}{\partial x \partial y} - \frac{\partial^2 \tau_{x\eta}}{\partial y^2}\} - (\frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial \tau_{x\eta}}{\partial y})(\frac{R^2}{4\Omega^2\ \cos^2\theta})\frac{2\Omega}{R^2}\ \sin\theta$$

$$= -\frac{R}{2\Omega\cos\theta}\{\frac{\partial^2 \tau_{y\eta}}{\partial x \partial y} - \frac{\partial^2 \tau_{x\eta}}{\partial y^2}\} - (\frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial \tau_{x\eta}}{\partial y})(\frac{1}{2\Omega\ \cos\theta})\tan\theta$$

$$= -\frac{1}{2\Omega\ \cos\theta}\{R(\frac{\partial^2 \tau_{y\eta}}{\partial x \partial y} - \frac{\partial^2 \tau_{x\eta}}{\partial y^2}) + \tan\theta(\frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial \tau_{x\eta}}{\partial y})\}$$

Thus

$$\frac{\partial M_x}{\partial x} = -\frac{1}{2\Omega\ \cos\theta}\{R(\frac{\partial^2 \tau_{y\eta}}{\partial x \partial y} - \frac{\partial^2 \tau_{x\eta}}{\partial y^2}) + \tan\theta(\frac{\partial \tau_{y\eta}}{\partial x} - \frac{\partial \tau_{x\eta}}{\partial y})\}$$

Integration of the right-hand side is not trivial, as the wind stress-vector components $\tau_{x\eta}$ and $\tau_{y\eta}$ could depend on the variable x besides y.

The global wind patterns on the surface (refer to the figures from the text) show that the x component of the wind stress vector is large than the y component. In other words the winds are mainly *zonal*; $\tau_{x\eta} >> \tau_{y\eta}$. If one can thus ignore the $\tau_{y\eta}$ term, the above equation becomes

$$\frac{\partial M_x}{\partial x} = \frac{1}{2\Omega\ \cos\theta}\{R\frac{\partial^2 \tau_{x\eta}}{\partial y^2} + \frac{\partial \tau_{x\eta}}{\partial y}\ \tan\theta\}$$

If we further assume that $\tau_{x\eta}$ is independent of x, the above equation yields

$$M_x = \frac{x}{2\Omega\ \cos\theta}\{R\frac{\partial^2 \tau_{x\eta}}{\partial y^2} + \frac{\partial \tau_{x\eta}}{\partial y}\ \tan\theta\} + f(y)$$

where $f(y)$ is the integration constant (independent of x). If there is a land mass at $x = a$, then $M_x = 0$ on $x = a$. Applying this boundary condition, we obtain for M_x

$$M_x = \frac{(x-a)}{2\Omega\ \cos\theta}\{R\frac{\partial^2 \tau_{x\eta}}{\partial y^2} + \frac{\partial \tau_{x\eta}}{\partial y}\ \tan\theta\}$$

and M_y in this case is

$$M_y = \frac{1}{\beta}\ \mathrm{Curl}_z\vec{\tau}_\eta = -\frac{R}{2\Omega\cos\theta}\frac{\partial \tau_{x\eta}}{\partial y}$$

Please refer to the text book for sketch of "streamlines" corresponding to $\vec{M} = M_x\hat{\imath} + M_y\hat{\jmath}$ for above solution. Also, read the section on the occurrence of the counter-current in the equatorial Pacific which is attributed to the variability of the wind stress, specifically to the *doldrums* in the trade winds, and that it can be predicted by the Sverdrup's model.