Lecture #12

Net Mass Transport by an Ekman Current

Of more practical significance than the velocity field in a wind-driven corrent is the knowledge of the rate of mass (water) transport in a wind-driven current. Based on the Ekman solution, given in the previous lecture, one can integrate to determine the rate of mass transfer across a control surface of unit width and of height extending from the free surface to the bottom:

$$M_x \equiv \int_{-\infty}^{0} \rho u \ dz; \quad M_y \equiv \int_{-\infty}^{0} \rho v \ dz.$$

However, one can by cleverly manipulating the governing equations obtain these expressions in a straightforward manner. Recall, that Ekman's equations are given by

$$0 = f v_E + A_z \frac{d^2 u_E}{dz^2}$$

$$0 = -f u_E + A_z \frac{d^2 v_E}{dz^2}$$

As $\frac{\partial w_E}{\partial x} = \frac{\partial w_E}{\partial y} = 0$, because w_E is small and the horizontal length scales large,

$$\tau_{xz} = \rho A_z (\frac{\partial u_E}{\partial z} + \frac{\partial w_E}{\partial x}) \approx \rho A_z \frac{du_E}{dz} \ \rightarrow \ \frac{du_E}{dz} = \frac{1}{\rho A_z} \tau_{xz}$$

$$\tau_{yz} = \rho A_z (\frac{\partial v_E}{\partial z} + \frac{\partial w_E}{\partial y}) \approx \rho A_z \frac{dv_E}{z} \ \rightarrow \ \frac{dv_E}{dz} = \frac{1}{\rho A_z} \tau_{vz}$$

One can then write the above momentum equations as

$$0 = \rho f v_E + \frac{d\tau_{xz}}{dz}$$

$$0 = -\rho f u_E + \frac{d\tau_{yz}}{dz}$$

Integrating the above equations with respect to z from $-\infty$ to 0, noting the definitions for M_x and M_y, and satisfying the continuity of shear stress across the sea surface, we obtain

$$0 = \int_{-\infty}^{0} \rho f v_E dz + \int_{-\infty}^{0} \frac{d\tau_{xz}}{dz} dz \ \rightarrow 0 = f M_{yE} + \tau_{x\eta} \ \rightarrow M_{yE} = -\frac{\tau_{x\eta}}{f}$$

and

$$0 = \int_{-\infty}^{0} -\rho f u_E dz + \int_{-\infty}^{0} \frac{d\tau_{yz}}{dz} dz \ \rightarrow 0 = -f M_{xE} + \tau_{y\eta} \ \rightarrow M_{xE} = \frac{\tau_{y\eta}}{f}$$

where $\tau_{x\eta}$ and $\tau_{y\eta}$ denotes x and y components of the wind-stress vector $\vec{\tau}_\eta$:

$$\vec{\tau}_\eta = \tau_{x\eta} \hat{i} + \tau_{y\eta} \hat{j}$$

What is striking in these expressions is the fact that the mass transport along the x direction is governed by the y component of the wind-stress vector, and *vice versa*! It is also practically significant to note that the mass transport of water in the wind-driven current can be estimated based on the knowledge of wind-stress and location (to know f). in other words, even without the knowledge of velocity distribution in the ocean current driven by wind, one can estimate the direction and amount of mass transport by simply knowing $\vec{\tau_\eta}$ and f; *ain't* it awesome?!

One can formally show that the direction of mass transport is normal to that of the wind stress in the following manner. Let us define rate of mass transport vector as

$$\vec{M}_E = M_{xE}\hat{i} + M_{yE}\hat{j}$$

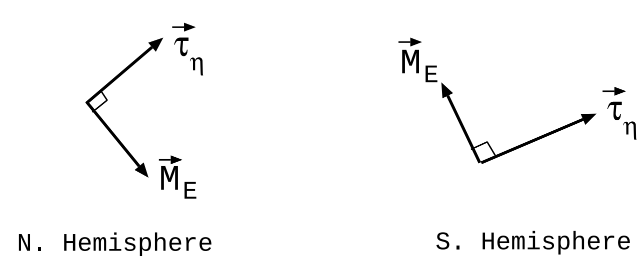
Taking the dot product of $\vec{\tau_\eta}$ and $\vec{M}_E$, and noting that $M_{xE} = \tau_{y\eta}/f$ and $M_{yE} = -\tau_{x\eta}/f$, we observe

$$\vec{\tau_\eta} \cdot \vec{M}_E = \tau_{x\eta}M_{xE} + \tau_{y\eta}M_{yE} = \tau_{x\eta}\frac{\tau_{y\eta}}{f} + \tau_{y\eta}(-)\frac{\tau_{x\eta}}{f} = 0$$

which means that the direction of mass transport is perpendicular to the direction of the wind! By taking their cross product, we can be more specific:

$$\vec{M}_E \times \vec{\tau_\eta} = M_{xE}\tau_{y\eta} - M_{yE}\tau_{x\eta} = \frac{\tau_{y\eta}}{f}\tau_{y\eta} + \frac{\tau_{x\eta}}{f}\tau_{x\eta} = \frac{|\vec{\tau_\eta}|^2}{f}$$

which is positive(negative) in the northern(southern) hemisphere because of the sign of f. The cross product $\vec{M}_E \times \vec{\tau_\eta}$ positive(negative) means, per right-hand coordinates used, that $\vec{M}_E$ is to the right(left) of $\vec{\tau_\eta}$. In other words, the direction of the mass transport is perpendicular and to the right of the wind stress in the northern hemisphere. The direction of the mass transport is perpendicular and to the left of the wind stress in the southern hemisphere (see figure below).



Conjectures on upwelling and downwelling near the coast (because of the presence of land mass) or in the open ocean (because of variability in the wind stress) based on the Ekman solution, will be presented in class. Note that these can only be conjectures, as the Ekman model does not account for the presence of lateral boundary nor does it allow variability of the wind stress in space.