Lecture #11

VIII. Wind-Driven Current - Ekman Solution

In this chapter we shall discuss ocean surface currents driven by wind. Transfer of momentum from wind to water could occur because of pressure fluctuations in the atmosphere associated with the wind or by viscosity shear across the interface, and it is the latter that is the consideration of this chapter. The exact problem of wind-driven current is extremely difficult one to analyze as it requires the solution coupled two-layer (air and sea) model governed by Navier-Stokes equations and nonlinear boundary conditions. In the present chapter, we shall only discuss the classical model of Ekman who obtained closed-form solution after making a series of assumptions. Despite the limitations implicit by the assumptions, the Ekman solution does capture some interesting and counter-intuitive behavior of wind-driven currents.

Ekman made the following assumptions in order to simplify the Navier-Stokes equations:

1. The problem is treated as a one-layer fluid flow. The wind stress is assumed specified on the free-surface. Without loss of generality, let the wind stress $\tau_{y\eta}$ be along the y direction.
2. The flow is steady and horizontal. In other words $w = 0$ and $\frac{\partial}{\partial t}u,\ v,\ p = 0$.
3. The free-surface is flat and horizontal.
4. The ocean is infinitely deep.
5. There are no lateral boundaries.
6. The variation of velocity in the vertical direction is much larger than that along any horizontal direction; in other words, $\frac{\partial}{\partial z}(u, v) >> \frac{\partial}{\partial x,y}(u, v)$. As the flow is also assumed steady, $u = u(z),\ v = v(z)$ only.
7. Water density is constant. As the free-surface is also assumed to be flat, horizontal pressure gradient is zero.
8. The coefficient of eddy viscosity is constant. In a typical wind-driven current the coefficient of eddy viscosity is O(10^{-3} [m^2/s]) which is larger that molecular vicosity coefficient ν which

is O(10^{-6} [m^2/s]). The molecular diffusion can therefore be ignored.

Under the above assumptions, the leading-order terms of the Navier-Stokes equations become

$$fv_E + A_z \frac{d^2 u_E}{dz^2} = 0 \tag{1}$$

$$-fu_E + A_z \frac{d^2 v_E}{dz^2} = 0 \tag{2}$$

$$-\frac{\partial p}{\partial z} - \rho g = 0$$

where the subscript E is used to denote that above equations represent the Ekman model.

As density is assumed constant and as gage pressure is zero on the sea surface $z = \eta = 0$, the z-component of the momentum equation can be integrated to get

$$p = -\rho g z$$

The equations for horizontal components of velocity field (u_E, v_E) are coupled second-order ordinary differential equations. One can decouple these equation, by differentiating each of the equations with respect to z and using the other equation to get rid of the other unknown. In other words, differentiating Eq.(1) with respect to z twice and using Eq.(2) we obtain

$$\frac{d^4 u_E}{dz^4} + \frac{f^2}{A_z^2} u_E = 0 \tag{3}$$

And similarly by differentiating Eq.(1) with respect to z twice and using Eq.(2) we obtain

$$\frac{d^4 v_E}{dz^4} + \frac{f^2}{A_z^2} v_E = 0 \tag{4}$$

each of which is an explicit fourth-order ordinary differential equation.

Integration of Eqs. (3) and (4) will introduce four constants each which can be determined using the boundary conditions. Note that the original system of equations (1 and 2) is only second order for u_E and v_E. Therefore, we may expect only four "natural" boundary conditions. The equations for u_E and v_E are made into fourth-order while de-coupling the equations. One has to therefore "derive" four additional boundary conditions using Eqs.(1) and (2) and the natural four boundary conditions. The four natural boundary conditions are

$$u_E \to 0, \text{ as } z \to -\infty \tag{5}$$

$$v_E \to 0, \text{ as } z \to -\infty \tag{6}$$

$$\rho A_z \frac{du_E}{dz} = \tau_{x\eta} = 0, \text{ on } z = 0 \tag{7}$$

$$\rho A_z \frac{dv_E}{dz} = \tau_{y\eta}, \text{ on } z = 0 \text{ (wind stress specified to be along y direction)} \tag{8}$$

Note that Eqs.(7) and (8) represent the continuity of the shear stress across the air-sea interface.

Additional four boundary conditions can be obtained by manipulating the original system of equations given by Eqs.(1) and (2) and the above four natural boundary conditions (Eqs. 5 to 8). Combining Eqs.(5) and (6) with Eqs. (1) and (2) we get

$$\frac{d^2 u_E}{dz^2} \rightarrow 0, \text{ as } z \rightarrow -\infty \tag{9}$$

$$\frac{d^2 v_E}{dz^2} \rightarrow 0, \text{ as } z \rightarrow -\infty \tag{10}$$

Next, by differentiating Eq.(1) and Eq.(2) with respect to z once and combining with boundary equations (7) and (8), we obtain

$$\frac{d^3 u_E}{dz^3} = -\frac{\tau_{y\eta}}{\rho f} \text{ on } z = 0 \tag{11}$$

$$\frac{d^3 v_E}{dz^3} = 0 \text{ on } z = 0 \tag{12}$$

Keep in mind that the wind stress is assumed to be along the y direction and assumed specified. The problem of solving the two fourth-order ordinary differential equations (3) and (4), subject to boundary conditions (Eqs. 5 to 12) is a straightforward. Seeking the solution of the form

$$u_E = Ae^{kz}, \quad v_E = Be^{kz}$$

where A and B constants and k is complex and using the boundary conditions one can obtain the following as the solution for u_E and v_E:

$$\begin{aligned} u_E &= +U_o \cos(\frac{\pi}{4} + \frac{\pi}{D_E} z)\ e^{\frac{\pi}{D_E} z} \text{ for northern hemisphere} \\ u_E &= -U_o \cos(\frac{\pi}{4} + \frac{\pi}{D_E} z)\ e^{\frac{\pi}{D_E} z} \text{ for southern hemisphere} \\ v_E &= +U_o \cos(\frac{\pi}{4} + \frac{\pi}{D_E} z)\ e^{\frac{\pi}{D_E} z} \text{ for either hemisphere} \end{aligned}$$

where

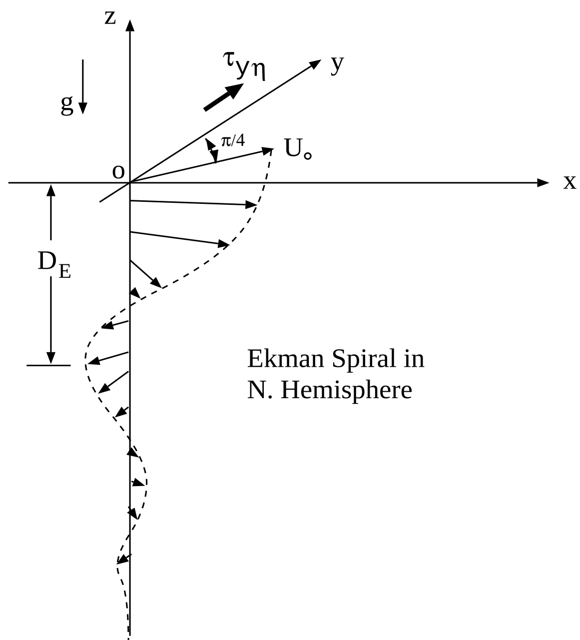
$$U_o = \frac{\sqrt{2}\ \pi\ \tau_{y\eta}}{D_E\ \rho\ |f|}$$

is called the surface current, which is nothing but $\sqrt{u_E^2 + v_E^2}$ on $z = 0$;

$$D_E = \pi \sqrt{\frac{2\ A_z}{|f|}}$$

is called the Ekman Depth, and f of course the Coriolis parameter defined as $2\Omega \sin\theta$.

The velocity profile, known as the Ekman Spiral, given by the above solution is sketched in following figure for the case of the current occuring in the northern hemisphere.



The following can be observed from the Ekman solution:

- On the surface, the current is 45^o to the right (left) of the wind-stress direction in the northern(southern) hemisphere. We shall show in the next section that the direction of net mass transport is perpendicular to the direction of the wind stress!

- At the Ekman depth, $z = -D_E$, the direction of the current is opposite to that on the surface U_o. In magnitude, the current speed is only $e^{-\pi} \approx 0.04$ times that of U_o. Therefore, for all proctical purposes, one can take the Ekman depth D_E to be the depth of influence of the wind. For a typical wind speed of $O(10[m/s])$ the Ekman depth is $(100[m])$ only.

- Even though the solution was obtained assuming that the wind-stress direction is along the y direction, one can take the solution as relative to any wind-stress direction; in other words, for example, if the wind-stress direction is towards east then the direction of the surface current is towards south-east or north-east depending on whether the location of the current is in the northern or southern hemisphere, respectively.