Lecture #10

## VII. Geostrophic Current

An important mechanism governing oceanographic flows is *geostrophy*, one in which pressure-gradient force balances the Coriolis force. These are the dominant forces for flows in which both Rossby and Ekman numbers are small; in other words, both inertia and viscous forces are small compared to the Coriolis force. Some typical characteristics of geostrophic currents in the ocean are that they are of large length scale (spanning almost the entire ocean) and slow with velocity of the order of magnitude O(0.1 [m/s]). In the vector form, the equation of motion for a geostrophic current is given by

$$-\nabla(p + \rho g z) - \rho 2\vec{\Omega} \times \vec{u} = 0$$

The pressure-gradient force, which balances the Coriolis force, can arise because of the slope of the sea surface, in which case the geostrophic current is called a *barotropic current*, and/or because of density variations in which case it is called a *baroclinic current*.

Based on order-of-magnitude analysis, one can determine the leading order equations governing an oceanographic current of geostrophic form. We shall use the following length and velocity scales, as well as assumptions, to reduce the Navier-Stokes equations:

- The horizontal length scale $L = 10^6$ [m]
- The vertical length scale $H = 10^3$ [m]
- The horizontal velocity scale $U = 10^{-1}$ [m/s]
- The time scale $T = L/U = 10^7$ [s]
- The vertical velocity scale $W = H / T = 10^{-4}$ [m/s]
- Flow occurs in the bulk of the ocean away from boundaries; therefore, viscosity effects are negligible.

Please refer to the previous lecture on physical characteristics of oceans and realize that the above length scales correspond to that of an entire ocean. Using the above characteristic values, and by order-of-magnitude estimate of each of the terms in the governing equations, we observe the following:

Equation of Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{U}{L} + \frac{U}{L} + \frac{W}{H}$$

As each term is of the same order of magnitude of $10^{-7}$ [$\mathrm{s}^{-1}$], all the terms have to be retained.

x-component of the momentum equation

In the following equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - f_y w + (\nu + \nu_T)\nabla^2 u$$

the intertia terms are all $\mathrm{O}(10^{-8}$ [$\mathrm{m/s^2}$]), the viscous and turbulent diffusion terms are negligible by assumption and the coriolis term $f_y w$ is of $\mathrm{O}(10^{-8}$ [$\mathrm{m/s^2}$]), while the coriolis term $fv$ is $\mathrm{O}(10^{-5}$ [$\mathrm{m/s^2}$]). The term $fv$, which is three orders of magnitude larger than the inertia term and $f_y w$, must therefore be balanced by the pressure gradient term $-\frac{1}{\rho}\frac{\partial p}{\partial x}$ Thus to the leading order, the x-component of the momentum equation becomes

$$0 = -\frac{\partial p}{\partial x} + \rho f v$$

Similarly, one can show that to the leading order, the y- and z-components of the momentum equations become

$$0 = -\frac{\partial p}{\partial y} - \rho f u$$

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

Note that the z-component of the momentum equation reduces to the familiar hydrostatic pressure equation. The leading-order geostrophic flow equations for $u, v, w$ and $p$ are thus given by

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
-\frac{\partial p}{\partial x} + \rho f v &= 0 \\
-\frac{\partial p}{\partial y} - \rho f u &= 0 \\
-\frac{\partial p}{\partial z} - \rho g &= 0
\end{aligned}$$

We shall study some interesting properties of a geostrophic flow by examining the above equations, and also solve these equations and determine flow velocities and directions for the case of homogeneous and inhomogeneous waters, in the following sections.

## Properties of a Geostrpohic Flow

We shall demonstrate some fascinating and counter-intuitive properties of a geostrophic flow using the leading-order geostrophic-flow equations obtained earlier, as follows. Let us define the horizontal velocity as

$$\vec{U} = u\hat{\imath} + v\hat{\jmath}$$

and the leading-order Coriolis force as

$$\vec{C} = C_x\hat{\imath} + C_y\hat{\jmath} = \rho f v\hat{\imath} - \rho f u\hat{\jmath}$$

(see the geostrophic-flow equations for $C_x$ and $C_y$ to be $\rho f v$ and $-\rho f u$, respectively. Taking the inner (dot) product of $\vec{C}$ and $\vec{u}$ we get

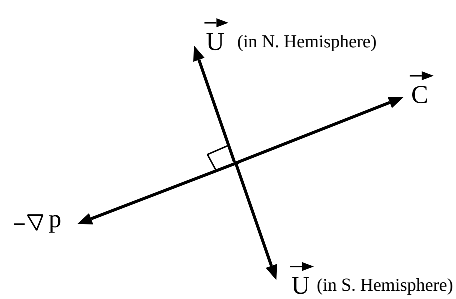
$$\vec{C} \cdot \vec{U} = \rho f v.u - \rho f u.v = 0$$

which means that [1] The flow direction is perpendicular to the Coriolis force.

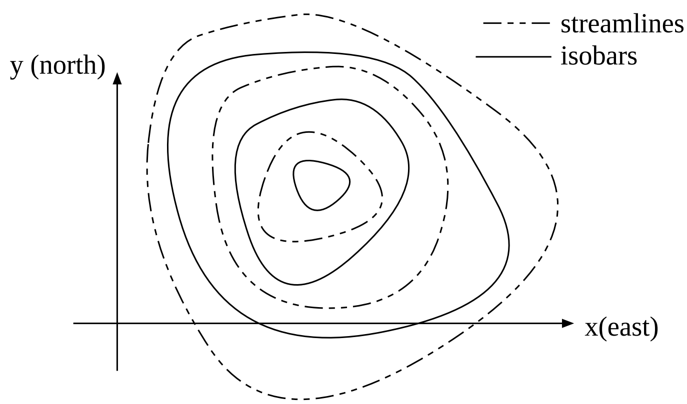
Next, consider the cross (vector) product of $\vec{C}$ and $\vec{U}$:

$$\vec{C} \times \vec{U} = \hat{k}(C_x v - C_y u) = \hat{k}\rho f(v^2 + u^2)$$

which is positive for positive $f$, which is the case in Northern Hemisphere, and is negative for negative $f$ (southern hemisphere). Per right-hand coordinates used in the problem formulation, positive $\vec{C} \times \vec{U}$ would imply that $\vec{C}$ is to the right of $\vec{U}$. If the cross product is negative, then $\vec{C}$ is to the left of $\vec{U}$. We thus note that [2] In the northern hemisphere, the Coriolis force is perpendicular and to the right to the direction of the flow, and in the southern hemisphere, the Coriolis force is perpendicular and to the left to the direction of the flow. The following figure illustrates the directions of the forces and flow in a geostrophic current.



Next let us study the relation between the contours on which the pressure is constant (also known as **isobars**) and the **streamlines** (which are tangent lines to the velocity-vector field), in a **horizontal plane** (ie. z=constant surface) for the case of a **steady geostrophic flow**. For illustration, consider the figure below in which isobars are denoted by solid lines and streamlines by broken lines.




On an isobar, pressure is constant and therefore

$$dp = 0$$

From basic calculus,

$$dp = \frac{\partial p}{\partial x}\, dx + \frac{\partial p}{\partial y}\, dy + \frac{\partial p}{\partial z}\, dz + \frac{\partial p}{\partial t}\, dt \;= 0$$

On the horizontal plane, z = constant and therefore $dz = 0$. For steady flow, as assumed above, $\frac{\partial p}{\partial t} = 0$. Therefore, on an isobar,

$$dp = 0 = \frac{\partial p}{\partial x}\, dx + \frac{\partial p}{\partial y}\, dy$$

Or, the slope of the isobar

$$\frac{dy}{dx} = -\frac{\partial p/\partial x}{\partial p/\partial y}$$

Since the geostrophic-flow equations, $\frac{\partial p}{\partial x} = \rho f v$ and $\frac{\partial p}{\partial y} = -\rho f u$,

$$\frac{dy}{dx}|_{\text{isobar}} = -\frac{\rho f v}{-\rho f u} = \frac{v}{u}$$

which is the same as that of a streamline (which is tangent to velocity vector $\vec{U} = u\hat{i} + v\hat{j}$). In other words, in the case of a <u>steady</u> geostrophic flow, the isobars and streamlines are one and the same!

Of course, to predict whether the flow is one way or the opposite on an isobar, one has to consider the facts that pressure gradient force balances the Coriolis force and that the Coriolis force is to the right (left) of the flow in the northern (southern) hemisphere.

## Geostrpohic Flow in a Homogeneous Ocean: Barotropic Current

Next, let us solve the geostrophic flow equations for the case in which the water is homogeneous flow is steady. In this case, the horizontal pressure gradient is due to the slope of a sea surface. The ensuing geostrphic current is called barotropic or slope current.

For $\rho =$ constant, which is the case for homogeneous ocean and steady flow, the z-component of the momentum equation

$$\frac{\partial p}{\partial z} = -\rho g$$

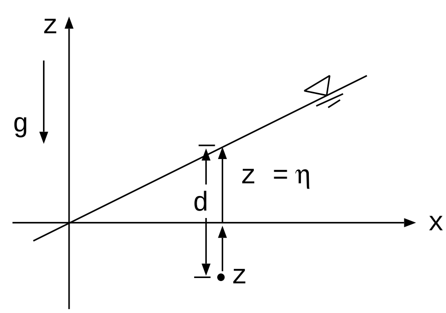
can be integrated to get

$$p = -\rho g z + C,$$

where $C$ is the integration constant. Expressing $p$ as gage pressure, in which case $p = 0$ on the sea surface $z = \eta$, one can find $C$ as

$$C = \rho g \eta,$$

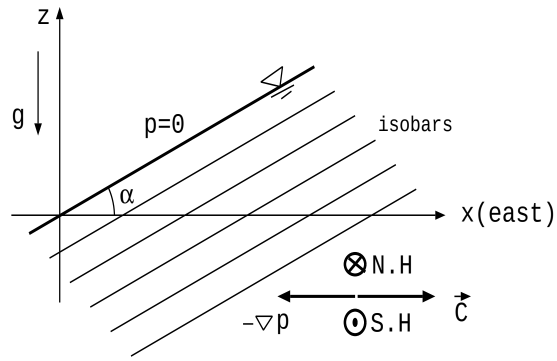
where $\eta$ denotes the sea surface elevation (see figure below).




Therefore,

$$p = -\rho g(z - \eta) \equiv \rho g d,$$

where $d$ is the depth measured from the actual free surface. Therefore, in this case, the isobars will be all parallel to the sea surface! (see figure below).




Let the local angle of inclination of the isobars (or the sea surface) be $\alpha$ towards east (x direction). On an isobar, $p$ = constant and therefore

$$dp = \frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz + \frac{\partial p}{\partial t}\,dt = 0$$

As we are considering the slope along east (in other words, on y = constant plane), $dy = 0$; as flow is assumed steady, $\frac{\partial p}{\partial t} = 0$. Therefore, the above equation becomes

$$dp = 0 = \frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial z}\,dz = 0$$

Re-arranging terms and noting that $\frac{dz}{dx} = \tan\alpha$, we get

$$\frac{dz}{dx}|_{\text{isobar}} \equiv \tan\alpha = -\frac{\partial p/\partial x}{\partial p/\partial z}$$

From the geostrophic-flow equations, $\partial p/\partial x = \rho f v$ and $\partial p/\partial = -\rho g$, and therefore the above equation becomes

$$\tan\alpha = -\frac{\rho f v}{-\rho g}$$

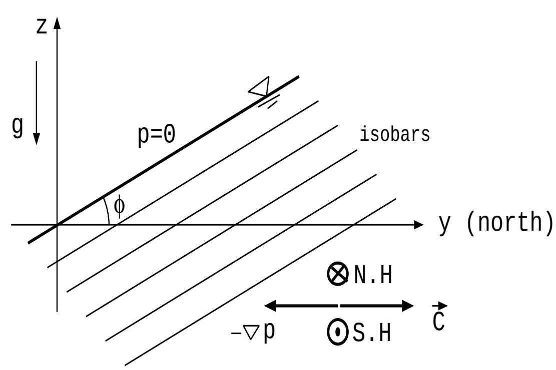
or,

$$v = \frac{g}{f}\,\tan\alpha$$

For positive slope along east (ie., tan $\alpha > 0$) and in the northern hemisphere (where $f$ is also positive), the velocity $v$ is positive (ie, towards north) as indicated by a cross-in-a-circle in the above figure. Note that in the figure, the y axis is into the paper. The cross in a circle symbolically represents the fin of an arrow which is what we will see as the arrow goes in the y direction (nice story, uh?!). On the other hand, for positive slope of the sea surface towards east and in the southern hemisphere (where $f$ is negative), $v$ will be negative, as denoted by the dot-in-a-circle. In the figure, the negative y axis is out of the paper, and the dot-in-a-circle denotes the tip of the arrow which is what one will see as it shoots out. In other words, the flow will be along the negative y direction in the southern hemisphere.

Figuratively also the above finding makes sense. As the sea surface is sloped positive towards east (x direction), the pressure-gradient force on a level (z=constant) surface will be along the negative x direction. The pressure gradient force will be balanced by the Coriolis force in a geostrophic flow, and hence the latter will be along the positive x direction. We know that in the northern hemisphere, the Coriolis force is normal and to the right of the flow; therefore, the current is normal and into the paper. Similarly, if the flow occurs in the southern hemisphere the Coriolis force is normal and to the left of the flow direction; hence, the current will be southward (out of the paper).

One can similarly obtain the relation for the x component of the velocity, u. For that, consider the figure of sea surface and isobars on the yz plane as shown below.




Here, as before, on an isobar,

$$dp = \frac{\partial p}{\partial y}\, dy + \frac{\partial p}{\partial z} = 0$$

as $x$ = constant and flow is steady. Using the geostrophic-flow equations, and re-arranging terms, we obtain for the slope of an isobar

$$\frac{dz}{dy}|_{\text{isobar}} \equiv \tan\phi = -\frac{\partial p/\partial y}{\partial p/\partial z}$$

From the geostrophic-flow equations, $\partial p/\partial y = -\rho f u$ and $\partial p/\partial z = -\rho g$; therefore,

$$\frac{dz}{dy}|_{\text{isobar}} \equiv \tan\phi = -\frac{-\rho f u}{-\rho g}$$

Thus we obtain for $u$

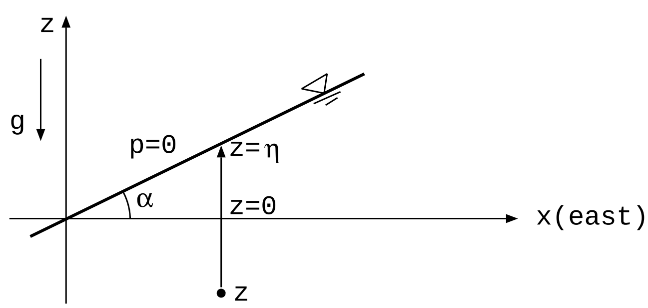
$$u = -\frac{g}{f}\ \tan\phi$$

where *phi* is the local angle of inclination towards north (y direction).

As done in the case for velocity component $v$, check to make sure that the above expression for $u$ is also consistent with the figure shown on the previous page and satisfies the properties of a geostrophic flow.

## Geostrpohic Flow in Inhomogeneous Ocean with a Sloped Surface: Barotropic and Baroclinic Currents

Let us next consider the case of a geostrophic flow occurring in an inhomogeneous ocean with a sloped sea surface. Here the pressure-gradient force arises because of variability of density (in which case the current is called a baroclinic current) and because of the surface slope (the corresponding current called the barotropic or slope current). The formal and proper derivation presented in this lecture notes was given by former student Mr. Stanislas van den Berg (MSOE in Acoustics 1998 and L' École Centrale, Paris, 1996) after a bitter discussion and argument in the class!.




As the pressure (gage) is zero on $z = \eta$, $\frac{\partial p}{\partial z} = -\rho g$, pressure at z can be written as

$$p = -\int_{\eta}^{z} \rho g dz = \int_{z}^{\eta} \rho g dz$$

Splitting the integration from $\eta$ to $z$ into (i) from $z$ to 0 and (ii) from 0 to $\eta$, we get

$$p = \int_{z}^{0} \rho g dz + \int_{0}^{\eta} \rho g dz$$

It is reasonable to assume that the water is well mixed near the surface, because of wave and other air-sea interaction processes. Let the density near the surface be $\rho_o$ which is a constant. The second integral can therefore be simplified and integrated:

$$\begin{aligned} p &= \int_z^0 \rho g dz + \rho_o g \int_0^\eta dz \\ &= \int_z^0 \rho g dz + \rho_o g \eta \end{aligned}$$

(note $g$ is also a constant, independent of z).

Differentiating the above expression for pressure with respect to x,

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \int_z^0 \rho g dz + \rho_o g \frac{\partial \eta}{\partial x}$$

Using the geostrophic-flow equations, $\partial p / \partial x = \rho f v$ and noting that $\partial \eta / \partial x \equiv \tan\alpha$ (slope of the sea surface), above equation can be written as

$$\rho f v = \frac{\partial}{\partial x} \int_z^0 \rho g dz + \rho_o g \frac{\partial \eta}{\partial x}$$

or

$$v = \frac{1}{\rho f} \frac{\partial}{\partial x} \int_z^0 \rho g dz + \frac{\rho_o}{\rho} \frac{g}{f} \tan \alpha$$

The first term on the right represents the velocity component due to density variations along the x direction and the second term the component driven by the sea-surface slope; the former is called *baroclinic* component and the latter the *barotropic* or slope component of the geostrophic current. If density is constant, then the above expression will reduce to $v = \frac{g}{f} \tan \alpha$ as obtained in the previous section.

Similarly, by considering the surface slope in the y direction and the y and z components of the geostrophic-flow equations, one can obtain for $u$:

$$\frac{\partial p}{\partial y} = -\rho f u = \frac{\partial}{\partial y} \int_z^0 \rho g dz + \rho_o g \frac{\partial \eta}{\partial y}$$

or

$$u = -\frac{1}{\rho f} \frac{\partial}{\partial y} \int_z^0 \rho g dz - \frac{\rho_o}{\rho} \frac{g}{f} \tan \phi$$

where $\tan \phi \equiv \frac{\partial \eta}{\partial y}$ denotes the slope of the surface in the y direction. Again, the first term on the right represents the baroclinic component and the second term the barotropic component of the geostrophic flow. For $\rho =$ constant, the above expression for $u$ reduces to the one obtained for homogeneous ocean in the previous section (ie., $u = -\frac{g}{f} \tan \phi$).