Lecture #8

V. Inertia Current

Having formulated the governing equations for a oceanographic flow, in an earth-fixed coordinates and including turbulence, we consider various types of oceanographic flows. Let us begin with a simple flow known as the inertia current. An inertia current refers to a horizontal flow in the absence of external forces and as observed in the earth-fixed frame of reference. The motion itself may have been initiated by a wind, but the motion will pursue even after the external force ceases to exist because of inertia. In this chapter, we shall derive expressions for inertia current paths using Lagrangian description of the flow.

The following assumptions are implicit and also explicitly made in the analysis of inertia currents:

- The current is horizontal (ie., $w = 0$)
- f-plane approximation (ie., the Coriolis parameter is assumed constant)
- pressure-gradient and viscous forces are all zero.

With the above assumptions, the equations of motion (horizontal components) for particle velocity can be reduced to

$$\frac{Du}{Dt} = fv \tag{1}$$

$$\frac{Dv}{Dt} = -fu \tag{2}$$

Above represent a pair of coupled, first-order, ordinary differential equations for particle-velocity components u and v. Upon decoupling, we obtain for u:

$$\frac{d}{dt}(\frac{du}{dt}) = \frac{d^2u}{dt^2} = f(\frac{dv}{dt}) = f(-fu) = -f^2u, \quad \rightarrow \frac{d^2u}{dt^2} = -f^2u \tag{3}$$

and similarly one can obtain by eliminating u in Eqn.(2) the following equation for v:

$$\frac{d^2v}{dt^2} = -f^2v \tag{4}$$

Integrating Eqn.(3), we obtain

$$u = A\ \sin ft + B\ \cos ft \tag{5}$$

where A and B are integration constants. Without loss of generality, let us set $u = 0$ at time $t = 0$. With this initial condition, Eqn.(5) becomes

$$u = A\ \sin ft \tag{6}$$

Substituting Eqn.(6) in Eqn.(1), we obtain

$$\begin{aligned} v &= \frac{1}{f}\frac{du}{dt} \\ &= \frac{1}{f}(Af\ \cos ft) \\ &= A\ \cos ft \end{aligned} \tag{7}$$

Denoting the speed of the current as U, we observe

$$U = \sqrt{u^2 + v^2} = \sqrt{A^2(\sin^2 ft\ +\ \cos^2 ft)} = A$$

that the constant A corresponds to the speed of the current. The velocity components u and v of a particle can therefore be expressed as

$$u = U\ \sin ft, \quad v = U\ \cos ft$$

Having determined u and v, one can determine the equations for a particle trajectory in an inertia current:

$$u = \frac{dX}{Dt} = U\sin ft\ \rightarrow X(t) = -\frac{U}{f}\ \cos ft + C_1 \tag{8}$$

and

$$v = \frac{dY}{Dt} = U\cos ft\ \rightarrow Y(t) = +\frac{U}{f}\ \sin ft + C_2 \tag{9}$$

where C_1 and C_2 are integration constants. Combining the above two equations, we observe

$$(X - C_1)^2 + (Y - C_2)^2 = (\frac{U}{f})^2 \tag{10}$$

which represents a circle with center at (C_1, C_2) and of radius $R = \frac{U}{|f|}$. The trajectory of inertia current is thus a cicle, known as the *inertia circle*. (Question: What is the trajectory of the inertia current, when observed from an *inertial* frame?!).

The period of inertia current (time required for a particle to execute one full circle) is equal to $2\pi R/U = 2\pi(U/|f|)/U = 2\pi/f$:

$$T = \frac{2\pi}{f} = \frac{2\pi}{|2\Omega\ \sin\theta|} = \frac{1\text{day}}{2\ |\sin\theta|}$$

where θ denotes the latitude. As shown earlier, the radius of the inertia circle is given by

$$R = \frac{U}{|f|}$$

From above expressions, we observe the following

- The radius of the inertia circle gets smaller and smaller as it occurs further and further away from the equator.
- On the equator, $R = \infty$! What does this mean?
- The period of the inertia current is smaller if it occurs farther away from the equator; it is 0.5 day at the poles, 1 day at 30^o.

The trajectory of the inertia circle is clockwise in the northern hemisphere and counter-clockwise in the southern hemisphere (can you prove this, using the equations and expressions derived in the class?!). The trajectory is also referred to as *cum sole* (or with the sun) for the inertia current in the northern hemisphere and *contra sole* (or against the sun) for the current occurring in the southern hemisphere. Being in the norhtern hemisphere and facing the equator, the motion of the sun would appear clockwise. The motion of the sun would be counter-clockwise to an observer in the southern hemisphere facing the equator. The motion of the inertia current is therefore *cum sole* if the current occurs in the northern hemisphere and *contra sole* if the current occurs in the southern hemisphere.

Please refer to standard descriptive texts on Physical Oceanography, such as *Ocean Circulation* published by the Open University, for field observations of inertia currents.