Lecture #7

IV. Non-Dimensional Parameters

Recall, in the vector form, the exact incompressible Navier-Stokes equations, governing oceanographic flows are given by

$$\nabla \cdot \vec{u} = 0$$

$$\rho[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}] = -\nabla p - \rho g\hat{k} - \rho 2\vec{\Omega} \times \vec{u} + \mu \nabla^2 \vec{u}$$

Note that the centrifugal acceleration (due to earth's rotation) has been absorbed in to gravity as a correction for magnitude. The relative importance of each of the external forces, compared to inertia, can be determined by scaling the terms with respective characteristic values and taking ratios with respect to one another. As a very simple example, let us say that characteristic speed of the flow is U and length L. Then the inertia force is of the order of $\rho U^2/L$ (which can be determined by considering the appropriate acceleration term, for e.g., $u \frac{\partial u}{\partial x}$. The viscous force is of the order of $\mu U/L^2$ which again can be found by considering the diffusion term (eg., $\mu \frac{\partial^2 u}{\partial x^2}$). The ratio of inertia to viscous force is then

$$\frac{\text{inertia}}{\text{viscous}} = \frac{\rho U^2/L}{\mu U/L^2} = \frac{UL}{\mu/\rho} = \frac{UL}{\nu}$$

which is called the Reynolds number!

$$\text{Reynolds Number, } Re \equiv \frac{UL}{\nu}$$

where $\nu \equiv \mu/\rho$ is the coefficient of kinematic viscosity. A low Reynolds number would mean that the viscous force is more dominant than inertia and a high Reynolds number would mean vice versa. Note the a turbulent flow is a high-Reynolds number flow.

Next, let us consider the importance of Coriolis force. The order of magnitude of the Coriolis force is $\rho\Omega U$ or $\rho f U$, where f $(= 2\ \Omega\ \cos\theta)$ is the Coriolis parameter. The ratio of inertia to Coriolis force is then

$$\frac{\text{inertia}}{\text{coriolis}} = \frac{\rho U^2/L}{\rho f U} = \frac{U}{fL}$$

The above non-dimensional parameter is called the Rossby number

$$\text{Rossby Number, } Ro \equiv \frac{U}{fL}$$

A high Rossby number flow means that inertia is dominant compared to Coriolis force. A small Rossby number means that Coriolis force is more dominant than the inertia force.

The ratio of viscous to coriolis forces is another significant parameter in Oceanography. The ratio of these forces is given by

$$\frac{\text{viscous}}{\text{coriolis}} = \frac{\mu U/L^2}{\rho f U} = \frac{\nu}{fL^2}$$

which is known as the Ekman number

$$\text{Ekman Number, } Ek \equiv \frac{\nu}{fL^2}$$

As we will see later, a flow in which both the Rossby number and Ekman number are small (both Ro, Ek $<< 1$) is referred to as a *Geostrphic flow*. In a geostrophic flow, the Coriolis force is balanced by the pressure-gradient force.

In flows in which the gravity force plays an important role, surface gravity waves for example, the relative importance of inertia and gravity forces is represented by *Froude number*. The ratio of these forces is given by

$$\frac{\text{inertia}}{\text{gravity}} = \frac{\rho U^2/L}{\rho g} = \frac{U^2}{gL}$$

The square-root of the above ratio is referred to as the Froude number:

$$\text{Froude Number, } Fn \equiv \frac{U}{\sqrt{gL}}$$

A small Froude number flow means that gravitational force is more dominant (eg, gentle smooth surface wave) and a large Froude number indicates that the inertia force is more significant than the restoring gravitational force (eg., breaking surface waves).

To summarize,

- Reynolds number, Re $\equiv \frac{UL}{\nu}$
- Rossby number, Ro $\equiv \frac{U}{fL}$
- Ekman number, Ek $\equiv \frac{\nu}{fL^2}$
- Froude number, Fn $\equiv \frac{U}{\sqrt{gL}}$