Lecture #6

III. Turbulence

Most oceanographic flows, or flows of nature in general, are turbulent. A turbulent flow in general is irregular and random. Some other essential features of a turbulent flow are

- Vortical and Three Dimensional: Turbulent flows are highly vortical and three-dimensional. Vortices (eddies) of a range of length scales are present, with the large ones (containing energy) corresponding to span of the flow domain and the small ones (at which energy is dissipated to heat by the action of viscosity) depending on the Reynolds number. Energy is transferred from large eddies to smaller and smaller eddies, by a nonlinear mechanism, upto the scale at which the energy is dissipated rapidly by viscosity.
- Dissipation: Large scale eddies draws energy by *shearing the mean flow* and transfer to small scales at which the mechanical energy is dissipated to heat. Sustaining a turbulent flow therefore requires a continuous work done on the fluid.
- Diffusion: Turbulence enhances mixing of mass, heat and momentum. For example, one enhances mixing of sugar and cream in coffee by stirring it causing turbulence. Even without stirring, sugar and cream would mix by molecular diffusion but it could take hours.
- Large Reynolds Number: Flow becomes turbulent at large Reynolds number ($Re \equiv Ul/\nu$). Flow is laminar (deterministic) at low Reynolds number.

Above are some essential properties of a turbulent flow. In this introductory course on Physical Oceanography, only rudimentary elements of the subject of turbulence will be discussed. The reader may want to refer to standard texts or recent articles to learn more about this difficult subject. [1] In this course, we will treat the subject of turbulence by deriving the classical Reynolds-averaged Navier-Stokes equations for the *mean flow* and modeling turbulence stresses using the notion of eddy viscosity.

[1] "I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightment. One is *quantum electrodynamics* and the other is the *turbulent motion of fluids*. And about the former I am rather optimistic" - Horace Lamb (1849-1934)

III-1 Reynolds Averaging of Navier-Stokes Equations

In most practical applications, one is interested mainly in the mean or average behavior of a flow and its dynamics and not in the exact and instantaneous behavior which is chaotic in a turbulent flow. For illustration, let us consider the time history of flow velocity as shown in the figure below. As shown, the instantaneous velocity component u can be decomposed into a mean $\bar{u}$ and fluctuation u'; it is the mean quantity $\bar{u}$ that is practical interest. One can extract the mean quantity $\bar{u}$ from the instantaneous u by time averaging u over a duration that is large compared to the time scale of fluctuating velocity component u'. In other words,

$$\bar{u} \equiv \frac{1}{T}\int_{t}^{t+T} u \; dt,$$

$$\bar{u'} \equiv \frac{1}{T}\int_{t}^{t+T} u' \; dt = 0$$

$$u = \bar{u} + u'$$

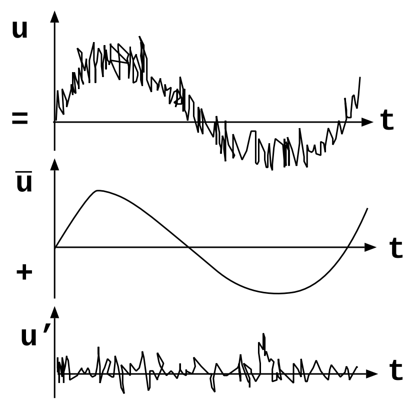

Fig 6.1 Decomposition of instantaneous velocity u into mean $\bar{u}$ and fluctuation u' components

The motivation here then is to obtain a set of equations for the mean quantities of velocity ($\vec{\bar{u}} = \bar{u}, \bar{v}, \bar{w}$) and pressure $\bar{p}$. One can obtain these equations by time averaging the governing exact Navier-Stokes equations (see previous lecture, p.25). While averaging, referred to as Reynolds averaging, the following rules (trivial proofs are left as exercise) are observed:

$$\begin{aligned} \overline{u+v} &= \bar{u} + \bar{v} \\ \overline{\rho u} &= \rho \bar{u} \quad \text{(where } \rho \text{ is constant)} \\ \bar{\bar{u}} &= \bar{u} \end{aligned}$$

$$
\begin{aligned}
\overline{uv} &= \overline{(\bar{u} + u')(\bar{u} + v')} \\
&= \overline{\bar{u}\bar{v}} + \overline{\bar{u}v'} + \overline{\bar{v}u'} + \overline{u'v'} \\
&= \bar{u}\bar{v} + \bar{u}\overline{v'} + \bar{v}\overline{u'} + \overline{u'v'} \\
&= \bar{u}\bar{v} + \bar{u}.0 + \bar{v}.0 + +\overline{u'v'} \\
&= \bar{u}\bar{v} + \overline{u'v'}
\end{aligned}
$$

Note that $\overline{u'v'} \neq \overline{u'}\,\overline{v'} = 0$. It will be zero only iff u and v are *uncorrelated* which is not always the case in a turbulent flow. The last rule tells us that Reynolds averaging a product of two flow variables for the mean quantities will also introduce a term involving fluctuations. As we will later, modeling of the latter is the most crucial and difficult aspect of turbulent-flow analysis.

Before we begin to average the Navier-Stokes equations, let us recast the convective acceleration terms as follows. The x-component of convective acceleration given by

$$
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}
$$

can be written equal to

$$
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + u\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\}
$$

as $\nabla \cdot \vec{u} = 0$ for an incompressible fluid. Expanding and regrouping terms one can show that the x-component of convective acceleration can be written as

$$
\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z}
$$

Similarly one can show that y- and z- components of convective accleration can be written as

$$
\frac{\partial vu}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial vw}{\partial z}
$$

and

$$
\frac{\partial wu}{\partial x} + \frac{\partial wv}{\partial y} + \frac{\partial ww}{\partial z},
$$

respectively. In CFD, is form of representation of the convective acceleration $(\vec{u} \cdot \nabla)\vec{u}$ is referred to as the *conservatiion form.* Recall then that the incompressible Navier-Stokes equations are given by

$$
\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
$$

$$
\rho\{\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} + \frac{\partial wu}{\partial z}\} = -\frac{\partial p}{\partial x} + \rho f v - \rho f_y w + \mu\nabla^2 u
$$

$$\rho\{\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial vv}{\partial y}+\frac{\partial wv}{\partial z}\}=-\frac{\partial p}{\partial y}-\rho fu+\mu\nabla^2 v$$

$$\rho\{\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial vw}{\partial y}+\frac{\partial ww}{\partial z}\}=-\rho g-\frac{\partial p}{\partial z}+\rho f_y u+\mu\nabla^2 w$$

Let us now begin to average the above equations. As the average of a sum is sum of averages, we can consider the averaging of the equations term by term. Note that the quantities ρ, g, f, f_y and μ are constants. As the averaging process is repetitive for most terms, I consider the averaging of only a few sample terms in the class, given below, and leave the rest to you as homework exercise:

$$\bar{\frac{\partial u}{\partial x}}=\frac{\partial \bar{u}}{\partial x}$$

because

$$\bar{\frac{\partial u}{\partial x}}\equiv\frac{1}{T}\int_t^{t+T}\frac{\partial u}{\partial x}\,dt=\frac{\partial}{\partial x}\frac{1}{T}\int_t^{t+T}u\,dt=\frac{\partial \bar{u}}{\partial x}$$

Similarly one can show, for example,

$$\bar{\nabla^2 u}=\nabla^2\bar{u},\ \bar{\frac{\partial p}{\partial x}}=\frac{\partial \bar{p}}{\partial x}\ \text{ etc.}$$

The averaging of the nonlinear terms representing convective acceleration requires some special attention as it introduces additional terms involving velocity fluctuations. For example,

$$\bar{\frac{\partial uv}{\partial y}}=\frac{\partial \bar{uv}}{\partial y}=\frac{\partial \bar{u}\bar{v}}{\partial y}+\frac{\partial \bar{u'v'}}{\partial y}$$

introduces a term involving velocity fluctuations u' and v'. Thus carrying out the averaging of all the terms, one can obtain the following Reynolds-averaged Navier-Stokes equations:

Equation of Continuity:

$$\nabla\cdot\vec{\bar{u}}\equiv\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0$$

ie., the mean velocity is also divergence-free.

x component of the momentum equation:

$$\rho\{\frac{\partial \bar{u}}{\partial t}+\frac{\partial \bar{u}\bar{u}}{\partial x}+\frac{\partial \bar{u}\bar{v}}{\partial y}+\frac{\partial \bar{u}\bar{w}}{\partial z}+\frac{\partial \bar{u'u'}}{\partial x}+\frac{\partial \bar{v'u'}}{\partial y}+\frac{\partial \bar{w'u'}}{\partial z}\}=-\frac{\partial \bar{p}}{\partial x}+\rho f\bar{v}-\rho f_y\bar{w}+\mu\nabla^2\bar{u}$$

Moving the fluctuating velocity terms to the right, we write the above equation as

$$\rho\{\frac{\partial \bar{u}}{\partial t}+\frac{\partial \bar{u}\bar{u}}{\partial x}+\frac{\partial \bar{u}\bar{v}}{\partial y}+\frac{\partial \bar{u}\bar{w}}{\partial z}+\}=-\frac{\partial \bar{p}}{\partial x}+\rho f\bar{v}-\rho f_y\bar{w}+\mu\nabla^2\bar{u}-\rho\frac{\partial \bar{u'u'}}{\partial x}-\rho\frac{\partial \bar{v'u'}}{\partial y}-\rho\frac{\partial \bar{w'u'}}{\partial z}$$

Similary, one can obtain the y and z components of the Reynolds-averaged Navier-Stokes equations:

y component of the momentum equation:

$$\rho\{\frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{v}\bar{u}}{\partial x} + \frac{\partial \bar{v}\bar{v}}{\partial y} + \frac{\partial \bar{v}\bar{w}}{\partial z} +\} = -\frac{\partial \bar{p}}{\partial y} - \rho f \bar{u} + \mu \nabla^2 \bar{v} - \rho \frac{\partial \bar{u'v'}}{\partial x} - \rho \frac{\partial \bar{v'v'}}{\partial y} - \rho \frac{\partial \bar{w'v'}}{\partial z}$$

z component of the momentum equation:

$$\rho\{\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{w}\bar{u}}{\partial x} + \frac{\partial \bar{w}\bar{v}}{\partial y} + \frac{\partial \bar{w}\bar{w}}{\partial z} +\} = -\frac{\partial \bar{p}}{\partial z} - \rho g + \rho f_y \bar{w} + \mu \nabla^2 \bar{w} - \rho \frac{\partial \bar{u'w'}}{\partial x} - \rho \frac{\partial \bar{v'w'}}{\partial y} - \rho \frac{\partial \bar{w'w'}}{\partial z}$$

Closure Problem: The above four equations were obtained for the purpose of determining mean quantities $\bar{u}$, $\bar{v}$, $\bar{w}$ and $\bar{p}$. But the averaging has lead to additional unknowns involving the correlations of velocity fluctuations. There are no additional and independent principles available to determine these terms. One may think of the energy equation, but that will in turn result in more unknowns upon averaging. This problem of ending up with more number of unknowns than the number of equations based on principles/laws is known as the *closure* problem. In practice, one has to resort to empirical models to determine these additional unknowns. It should be pointed out here that the original Navier-Stokes equations are exact and are valid irrespective of whether a flow is laminar or turbulent. Unfortunately, averaging these equations for mean quantities leads to additional unknowns.

The terms $-\rho \bar{u'u'}$, $-\rho \bar{u'v'}$, $\bar{u'w'}$, $\bar{v'v'}$, $\bar{v'w'}$, etc arising while averaging the convective acceleration terms and appearing on the right-hand side of the above momentum equations are referred to as the *Reynolds stress tensor* components. The gradient of these stress components, as written on the right-hand side of the above equations, represents the contribution of the fluctuating velocity components to the transport (turbulent diffusion) of mean momentum, just as in some sense, viscosity reprsents the contribution of molecular collisions to the transport of macroscopic momentum of a fluid.

Turbulent/Eddy Viscosity Just as the transfer of momentum by collisions of molecules is modeled in terms of velocity gradients and coefficient of viscosity, the Reynolds-stress components are modeled in terms of the gradient of mean velocity and turbulent (or eddy) viscosity coefficients. For example, in the index notaion, as

$$\bar{u_i'u_j'} = \nu_t\{\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\}$$

The coeffcient ν_t is called the turbulent (or eddy) viscosity coeffient. This coefficient is a flow property; in other words, unlike molecular viscosity ν, the eddy viscosity depends on flow parameters such as speed, domain geometry etc. While molecular viscosity of water $\nu = O(10^{-6}\ [m/s^2])$, the eddy viscosity, depending on the flow could be as large as $O(10^{-2}\ [m/s^2])$. In such highly-turbulent flows, the effect of molecular diffusion is rather insignificant. The eddy-viscosity coeffients can only be empirically determined (or through numerical simulation of particular flow). In the literature one will find more sophisticated modeling of turbulent stresses. For the present, we shall leave the discussion of turbulence and turbulence modeling at this point, but take it up again later in the chapter on *wind-driven currents* in deep and shallow waters, the flows in which turbulent diffusion of mean momentum is significant.

Again for convenience, from here onwards, we shall drop the overbar denoting the mean, with the understanding that the equations are for the mean quantities. The equations are summarized below:

Equation of Continuity

$$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

x-component of the momentum equation

$$\rho\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\} = -\frac{\partial p}{\partial x} + \rho f v - \rho f_y w + (\mu + \rho\nu_t)\nabla^2 u$$

y-component of the momentum equation

$$\rho\{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\} = -\frac{\partial p}{\partial y} - \rho f u + (\mu + \rho\nu_t)\nabla^2 v$$

z-component of the momentum equation

$$\rho\{\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\} = -\rho g - \frac{\partial p}{\partial z} + \rho f_y u + (\mu + \rho\nu_t)\nabla^2 w$$

where

$f \equiv 2\Omega\ \sin\theta$, the Coriolis parameter

$f_y \equiv 2\Omega\ \cos\theta$, and

ν_t the coefficient of kinematic turbulent (eddy) viscosity.