Lecture #5

We showed in the previous lecture that the exact equations, namely the incompressible Navier-Stokes equations, governing oceanographic flows and defined with respect to a earth-surface fixed rotating coordinate system $oxyz$, are given by

$$\nabla \cdot \vec{u} = 0 \text{ (equation of continuity of incompressible fluid)} \quad (1)$$

$$\rho[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}] = -\nabla p - \rho g \hat{k} + \mu \nabla^2 \vec{u} - \rho(\vec{\Omega} \times [\vec{\Omega} \times \vec{R}]) - \rho(2\vec{\Omega} \times \vec{u}) \quad (2)$$

In oceanographic applications, one will find various approximations, conventions and terminology are introduced to the equations. These are discussed below.

II-5. Treatment of Centrifugal Acceleration

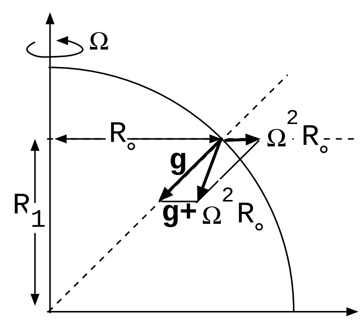

Fig 5.1 Centrifugal and Graviational accelerations.

Decomposing the postion vector $\vec{R}$ as

$$\vec{R} = \vec{R}_o + \vec{R}_1,$$

where $\vec{R}_1$ denotes the component parallel, and $\vec{R}_0$ the component perpendicular to the axis of rotation and substituting in the expression for centrifugal acceleration

$$-\vec{\Omega} \times [\vec{\Omega} \times (\vec{R}_1 + \vec{R}_0)]$$

one can show that the latter reduces to

$$-\vec{\Omega} \times [\vec{\Omega} \times \vec{R}_0]$$

Note the magnitude of angular velocity of the earth Ω is 2π/day $\approx 7 \times 10^{-5}$ [rad/s] and the radius of the earth is $\approx$ 6400 [Km]. The order of magnitude of the centrifugal acceleration $\Omega^2 R_o$ is therefore O(10^2 [m/s^2]), which is almost three orders of magnitude smaller than the acceleration of gravity which O(10 [m/s^2]). Figure 5.1 shows the directions of action of centrifugal and gravitational acceleration and also of their resultant. The direction of action of the resultant is called plumb line for a plumb line (or a stationary pendulum) will be directed along this line by the action of gravity and centrifugal accelerations. In oceanography, the centrifugal acceleration term is "absorbed" into the gravity term as the magnitude of the former is much smaller than that of the latter. Precisely, the centrifugal acceleration is taken as a correction to the gravitational acceleration ONLY in magnitude, but NOT in the direction. In other words the magnitude of the corrected g is that of the resultant of g and $\Omega^2 R_o$ while the direction is towards the center of the earth.

II-6. Convention for Earth-Fixed Coordinates and Components of Coriolis Acceleration

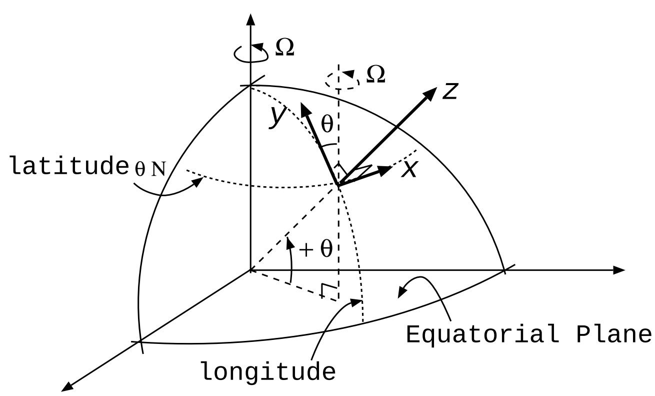

Fig 5.2 Earth-fixed coordinate system oxyz

As shown in Figure 5.2, the z-axis of the earth fixed coordinate system is taken to be radially outward (ie. locally upward), the x-axis toward the east and the y-axis toward the geometric north. The latitude θ, which is measured from the equator, is positive in the northern hemisphere and negative in the hemispere (to make it further clear (!), $\theta = 0$ deg on the equator, +90 deg at the north pole and -90 deg at the south pole).

The components of the angular velocity vector $\vec{\Omega}$ with respect to earth-fixed oxyz coordinate system are

$$\vec{\Omega} = 0\ \hat{\imath} + \Omega\ \cos\theta \hat{\jmath} + \Omega\ \sin\theta \hat{k}$$

The components of the Coriolis acceleration $-2\vec{\Omega} \times \vec{u}$ are therefore

$$-2\vec{\Omega} \times \vec{u} = \{2\Omega v\ \sin\theta - 2\Omega w\ \cos\theta\}\hat{\imath} - 2\Omega u\ \sin\theta\ \hat{\jmath} + 2\Omega u\ \cos\theta\ \hat{k}$$

The quantity 2 Ω sinθ, which is positive in the northern hemisphere and negative in the southern hemisphere (as sin is an odd function) plays an important role in oceanographic flows. The quantity is referred to as the *Coriolis parameter* and denoted as f:

$$\text{Coriolis Parameter, } f \equiv 2\Omega\ \sin\theta$$

In some papers and texts, the other component 2Ω cosθ is denoted as f_y, but its role is not as significant.

$$f_y \equiv 2\Omega\ \cos\theta$$

As discussed earlier, one uses a rectangular coordinate system for the formulation of oceanographic flows as the resulting equations are a lot simpler than the ones obtained in a spherical polar coordinate system, even though the latter is the obvious choice for studying very large flows spanning almost the entire planet. Even with the use of local rectangular coordinate system (Figure 5.2), the governing equations still pose a challenge as the Coriolis parameter f as well as f_y will be different for different fluid particles as it depends on θ. In Oceanography, as an approximation, the Coriolis parameter f is assumed to be constant over the entire horizontal xy plane that spans a particular flow. This approximation is known as the $f - plane$ approximation. As it turns out for most oceanographic flows, the *f-plane* approximation is reasonable for flows with horizontal length scale of O(100 [km]). In such flows, the Coriolis parameter f for entire region is assumed to be constant, which in value corresponds to that at the origin of oxyz taken somewhere preferably at middle.

The *f-plane* approximation is not justifiable for flows that span a horizontal distance of O(1000 [km]). Even for the analysis of such large horizontal-length scale flows, one finds the use of rectangular coordinate system but allowing some degree of variation of f over the region. Specifically, it what is referred to as the $\beta - plane$ approximation, the Coriolis parameter is Taylor-series expanded along the y-direction about the the origin, and local f determined using the first two terms. In other words,

$$\begin{aligned} f &= f_o + [\frac{\partial f}{\partial y}]_o\ y + [\frac{\partial^2 f}{\partial y^2}]_o + ... \\ &\approx f_o + [\frac{\partial f}{\partial y}]_o\ y = f_o + \beta y, \end{aligned}$$

where

$$\beta \equiv [\frac{\partial f}{\partial y}]_o$$

Please refer to the text for more on these and other standard conventions and approximations used in the treatment of oceanographic flows. Below, we summarize the governing equation, in component form, which will be frequently referred later while analyzing various oceanographic currents.

Oceanographic Flow Equations in Earth-Fixed Coordinates oxyz

Equation of Continuity

$$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

x-component of the momentum equation

$$\rho\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\} = -\frac{\partial p}{\partial x} + \rho f v - \rho f_y w + \mu \nabla^2 u$$

y-component of the momentum equation

$$\rho\{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\} = -\frac{\partial p}{\partial y} - \rho f u + \mu \nabla^2 v$$

z-component of the momentum equation

$$\rho\{\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\} = -\rho g - \frac{\partial p}{\partial z} + \rho f_y u + \mu \nabla^2 w$$

where

$f \equiv 2\Omega \sin\theta$, the Coriolis parameter

$f_y \equiv 2\Omega \cos\theta$, and

θ the angle of latitute measured from equator, positively on the northern hemisphere and negatively in the southern hemisphere.
