Lecture #4

II. Equations Governing Oceanographic Flows (contd.)

II-4. Equations of Motion with respect to a Rotating Frame of Reference

Recall from elementary dynamics that in the Newton's II law of motion, $\vec{F} = m\vec{a}$, $\vec{a}$ corresponds to the particle acceleration as observed in an *inertial* frame of reference. In may engineering applications, one assumes that an earth-fixed coordinats system is inertial and applies Newtons Laws directly to obtain equations of motion. In such applications, the contribution of the earth's rotation to inertia is negligibly small, that the assumption is justifiable. For example, in fluid mechanics problems such as mechanical pipe flow or ship hydrodynamics, it is perfectly alright to assume that the earth-fixed coordinate system used in the formulation to be inertial. However, in large scale oceanographic or atmospheric flows such as gyres and cyclones, the contribution of earth's rotation is significant that one can no longer assume the coordinates to be inertial. As it is convenient to study these flows using earth-fixed coordinate system, we need to relate the kinematics as observed in fixed inertial frame to that in non-inertial rotating frame in order to apply the Newtons Laws appropriately and obtain the equations of motion. Fortunately, the only term in the Navier-Stokes equations that need to be modified is the inertia term $\vec{a} = \frac{D\vec{u}}{Dt}$ which is presented in this lecture. The reader may want to review any stadard text on *Dynamics* for formal proofs involving *rate of change of a vector rotating frame of reference.*

Let us consider the planet earth rotating about the Y_o axis of the fixed inertial coordinates $OX_oY_oZ_o$ as shown in Fig. 4.1. Let the angular velocity of the earth $\vec{\Omega}$ be a constant. Let the second coordinate system $oxyz$, as shown in the figure, be fixed to the surface of the earth and hence a rotating frame of reference. We now compare the kinematics of a flow as observed in these two different coordinate system. Let $\vec{R}(t)$ denote the instantaneous position of a fluid particle from the fixed inertial $OX_oY_oZ_o$ coordinares and $\vec{r}(t)$ the position vector at the same instant of the same particle from the earth-fixed rotating frame of reference $oxyz$. Let $\vec{\rho}$ be the position vector of the origin of $oxyz$ with respect to $OX_oY_oZ_o$. Then by simple vector addition,

$$\vec{R} = \vec{\rho} + \vec{r} \tag{1}$$

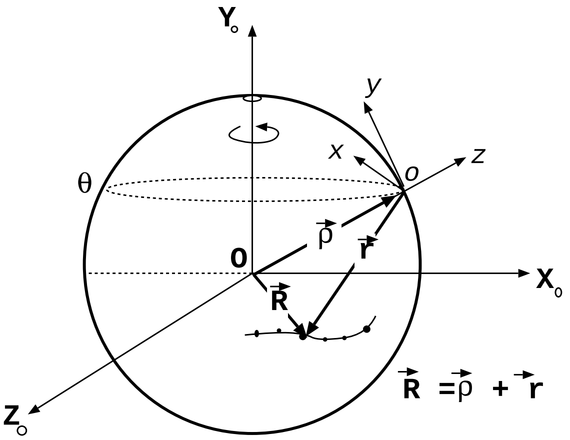

Fig 4-1. Earth fixed rotating coordinates $oxyz$ and fixed inertial coordinates $OX_oY_oZ_o$.

Differentiating Eq.(1) with respect to time, we obtain

$$\frac{d\vec{R}}{dt} = \frac{d\vec{\rho}}{dt} + \frac{d\vec{r}}{dt}, \tag{2}$$

where $\frac{d}{dt}$ denotes time rate of change as observed in the fixed inertial frame $OX_oY_oZ_o$. Recall from elementary kinematics, the this time rate of change is related to that observed in the rotating frame $oxyz$ (let us call it $\frac{d}{dt'}$) is given by

$$\frac{d}{dt} = \frac{d}{dt'} + \vec{\Omega}\times \tag{3}$$

where $\vec{\Omega}$ as said before denotes the angular velocity of the earth and it is assumed to be a constant. Therefore Eq.(2) becomes

$$\frac{d\vec{R}}{dt} = \frac{d\vec{\rho}}{dt'} + \vec{\Omega} \times \vec{\rho} + \frac{d\vec{r}}{dt'} + \vec{\Omega} \times \vec{r} \tag{4}$$

where $\frac{\vec{\rho}}{dt'} = 0$ as $\vec{\rho}$ is constant as oberved in the rotating frame. Therefore,

$$\frac{d\vec{R}}{dt} = \vec{\Omega} \times \vec{\rho} + \frac{d\vec{r}}{dt'} + \vec{\Omega} \times \vec{r} \tag{5}$$

In the above, $\frac{d\vec{r}}{dt'}$ denotes the rate of change of the position vector of the fluid particle (ie. fluid velocity) as observed in $oxyz$ which we shall denote henceforth as $\vec{u}$ with components being u, v, w. The left-hand side denotes the velocity as observed in the fixed inertial frame. Please make a note that from here onwards, $\vec{u}$ denotes fluid velocity as observed in the rotating frame of reference $oxyz$:

$$\frac{d\vec{r}}{dt'} \equiv \vec{u}$$

Equation (5) can therefore be written as

$$\frac{d\vec{R}}{dt} = (\vec{\Omega} \times \vec{\rho}) + \vec{u} + (\vec{\Omega} \times \vec{r}) \tag{6}$$

Differentiating the above with resect to time t again, we get (remember, $\vec{\Omega}$ is constant)

$$\frac{d^2\vec{R}}{dt^2} = (\vec{\Omega} \times \frac{d\vec{\rho}}{dt}) + \frac{d\vec{u}}{dt} + (\vec{\Omega} \times \frac{d\vec{r}}{dt}) \tag{7}$$

As $\frac{d}{dt} = \frac{d}{dt'} + \vec{\Omega}\times$ and $\frac{d\vec{\rho}}{dt'} = 0$, the above equation becomes

$$\begin{aligned} \frac{d^2\vec{R}}{dt^2} &= (\vec{\Omega} \times [\vec{\Omega} \times \vec{\rho}]) + (\frac{d\vec{u}}{dt'} + [\vec{\Omega} \times \vec{u}]) + ([\vec{\Omega} \times \frac{d\vec{r}}{dt'}] + \vec{\Omega} \times [\vec{\Omega} \times \vec{r}]) \qquad (8) \\ &= \frac{d\vec{u}}{dt'} + (\vec{\Omega} \times [\vec{\Omega} \times \{\vec{\rho} + \vec{r}\}]) + 2\vec{\Omega} \times \vec{u}, \quad \text{because } \frac{d\vec{r}}{dt'} \equiv \vec{u} \\ &= \frac{d\vec{u}}{dt'} + (\vec{\Omega} \times [\vec{\Omega} \times \vec{R}]) + 2\vec{\Omega} \times \vec{u}, \quad \text{because } \vec{R} = \vec{\rho} + \vec{r} \text{ see Figure 4.1} \end{aligned}$$

In the above equation, the left-hand side denotes the fluid acceleration as observed in the the fixed inertial frame $OX_oY_OZ_o$ and the first term on the right-hand side $\frac{d\vec{u}}{dt'}$ the acceleration as observed in the rotating frame of refernce $oxyz$ which henceforth will be denoted as $\vec{a}$. The term $(\vec{\Omega} \times [\vec{\Omega} \times \vec{R}])$ is called the centrepetal acceleration and the term $2\vec{\Omega} \times \vec{u}$ is called the Coriolis acceleration. With respect to the rotating frame of reference, the Newton's II law can therefore be written as

$$m\vec{a} + (\vec{\Omega} \times [\vec{\Omega} \times \vec{R}]) + 2\vec{\Omega} \times \vec{u} = \vec{F}$$

where $\vec{F}$ denotes the sum of the external forces (due to gravity, pressure and viscosity). Retaining only the inertia term (as observed in the rotating frame) on the left-hand side, the above can be written as

$$m\vec{a} = \vec{F} - m(\vec{\Omega} \times [\vec{\Omega} \times \vec{R}]) - m(2\vec{\Omega} \times \vec{u})$$

The term $-m(\vec{\Omega} \times [\vec{\Omega} \times \vec{R}])$ is called the *Centrifugal* force and $-m(2\vec{\Omega} \times \vec{u})$ is referred to as the *Coriolis* force. As you may recall from physics, these are not "actual" forces, but just resulting from inertia when defined in a rotating frame of reference.

The governing incompressible Navier-Stokes equations, defined with respect to a earth-surface fixed rotating coordinate system $oxyz$, is therefore (dropping the prime superscript $'$ for time rate of change in rotating frame)

$$\nabla \cdot \vec{u} = 0 \text{ (equation of continuity of incompressible fluid)} \tag{9}$$

$$\rho[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}] = -\nabla p - \rho g\hat{k} + \mu\nabla^2\vec{u} - \rho(\vec{\Omega} \times [\vec{\Omega} \times \vec{R}]) - \rho(2\vec{\Omega} \times \vec{u}) \tag{10}$$

More on these equations and their applications, next class.
