Lecture #3

II. Equations Governing Oceanographic Flows (contd.)

II-3. Balance of Linear Momentum in Viscous Fluid: The Incompressible Navier-Stokes Equation

Avoiding repetition of steps discussed earlier in the derivation of Euler's equation, we simply consider only the term that need to be modified to model effect of viscosity. As you may recall from basic fluid mechanics, viscosity is a macroscopic effect resulting from the transfer of momentum due to molecular collisions. Viscosity contributes to both normal and tangential components of surface force; therefore, in the case of a viscous flow, the surface force need not be normal (due to pressure) to the surface. The stress (surfae force per unit area) must therefore be expressed as a vector.

Stress Vector $\vec{\tau}$ and Stress Tensor $\tilde{\sigma}$

As you may recall from earlier courses in solid or fluid mechanics, stress tensor component, denoted as σ_{ij} represents force in the direction i on a surface with normal along direction j. For convenience, we use the indices. In the case of rectilinear coordinate system $i, j = 1, 2, 3$ would represent x, y, z respectively. You may also recall that the stress tensor σ_{ij} is symmetric; in other words $\sigma_{ij} = \sigma_{ji}$ which can be proved based on the principle of balance of *angular* momentum.

In the case of an incompressibe, linear viscous (Newtonian) fluid, the constitutive equation for the stress tensor is, in index notation, given by:

$$\sigma_{ij} = -p\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

where, per index notation, repeated indices denote summation and the symbol δ_{ij}, called the Kronecker Delta, represents 1 for $i = j$ and 0 otherwise. The term p represents the pressure field and the coefficient μ is called the coefficient of *dynamic* viscosity. The ratio μ/ρ, denoted as ν, is called the coefficient of *kinematic* viscosity. For water, $\nu \approx 10^{-6}$ $[m^2/sec]$.

The stress-tensor components, with respect to a coordinate system oxyz are therefore,

$$
\begin{aligned}
\sigma_{xx} &= -p + 2\mu\frac{\partial u}{\partial x} \\
\sigma_{yy} &= -p + 2\mu\frac{\partial v}{\partial y} \\
\sigma_{zz} &= -p + 2\mu\frac{\partial w}{\partial z} \\
\sigma_{xy} = \sigma_{yx} &= \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\
\sigma_{xz} = \sigma_{zx} &= \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \\
\sigma_{yz} = \sigma_{zy} &= \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})
\end{aligned}
$$

The force per unit area on a surface, called the stress vector $\vec{\tau}$, is related to the stress tensor $\tilde{\sigma}$ as follows (for proof, refer to a standard text on mechanics):

$$\vec{\tau} = \tilde{\sigma} \cdot \hat{n}$$

where $\hat{n}$ denotes the normal vector on the surface (see figure below).

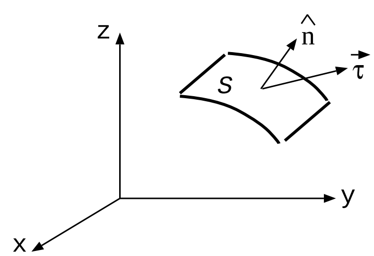

Fig 3-1. Normal and stress vectors on a surface S.

For the case of an incompressibe viscous flow, the stress-vector components (τ_x, τ_y, τ_z) are therefore

$$
\begin{aligned}
\tau_x &= \sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z \\
&= [-p + 2\mu\frac{\partial u}{\partial x}]n_x + \mu[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}]n_y + \mu[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}]n_z \\
\tau_y &= \sigma_{yx}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z \\
&= \mu[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}]n_x + [-p + 2\mu\frac{\partial v}{\partial y}]n_y + \mu[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}]n_z \\
\tau_z &= \sigma_{zx}n_x + \sigma_{zy}n_y + \sigma_{zz}n_z \\
&= \mu[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}]n_x + \mu[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}]n_y + [-p + 2\mu\frac{\partial w}{\partial z}]n_z
\end{aligned}
$$

Incompressible Navier-Stokes Equations

We start the presentation of the equation of motion of a viscous fluid from that obtained earlier for an inviscid fluid with pressure replaced by stress vector $\vec{\tau}$ (refer to Lecture # 2 also):

$$\int_\Omega \frac{\partial \rho \vec{u}}{\partial t}\, d\Omega + \int_S (\rho \vec{u})\vec{u}\cdot\hat{n}\; dS = -\hat{k}\;\int_\Omega \rho g\; d\Omega + \int_S \vec{\tau}\; dS.$$

As $\vec{\tau} = \tilde{\sigma}\cdot\hat{n}$, above equation can be rewritten as

$$\int_\Omega \frac{\partial \rho \vec{u}}{\partial t}\, d\Omega + \int_S (\rho \vec{u})\vec{u}\cdot\hat{n}\; dS = -\hat{k}\;\int_\Omega \rho g\; d\Omega + \int_S \tilde{\sigma}\cdot\hat{n}\; dS.$$

Application of the Gauss theorem to the surface integrals will result in

$$\int_\Omega \frac{\partial \rho \vec{u}}{\partial t}\, d\Omega + \int_\Omega \nabla\cdot(\rho \vec{u})\vec{u} = -\hat{k}\;\int_\Omega \rho g\; d\Omega + \int_\Omega \mathrm{div}\tilde{\sigma}\; d\Omega.$$

The differential form of the equation will therefore be

$$\frac{\partial \rho \vec{u}}{\partial t} + \nabla\cdot(\rho\vec{u})\vec{u} = -\rho g\hat{k} + \mathrm{div}\tilde{\sigma}$$

Expanding, and invoking the equation of continuity (as done in the derivation of the Eulers equation), we can reduce the above equation to

$$\rho[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot)\vec{u}] = \rho\frac{D\vec{u}}{Dt} = \mathrm{div}\tilde{\sigma} - \rho g\hat{k}$$

In the case of an incompressible linear-viscous fluid (water is an example) with the components of $\tilde{\sigma}$ given by $\sigma_{ij} = -p\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, one can show (exercise!) that

$$\mathrm{div}\tilde{\sigma} = -\nabla p + \mu\nabla^2\vec{u}$$

where ∇^2 represents the Laplacian operator (eg., in oxyz coordinates $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$).

The equation of motion for the incompressible linear-viscous fluid flow, referred to as the incompressible Navier-Stokes equations, are thus

$$\rho[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot)\vec{u}] = \rho\frac{D\vec{u}}{Dt} = -\nabla p + \mu\nabla^2\vec{u} - \rho g\hat{k}$$

In the case of an incompressible fluid, per conservation of mass,

$$\nabla\cdot\vec{u} = 0$$

The above two equations are for the unknowns of $\vec{u}$ and pressure p.