Lecture #2

II. Equations Governing Oceanographic Flows

Those in this class with a background in ocean, mechanical or civil engineering, may have already learnt the derivation of the equations governing fluid motions in their undergraduate courses in fluid mechanics, hydrology or wave mechanics. The present lecture is for the benefit of those with different background such as in electrical or computer engineering, curriculum of which may not contain a formal course in fluid mechanics. The present lecture could nevertheless serve as a review for those in the former category who have learnt the material in other previous courses.

II-1 Conservation of Mass; Equation of Continuity

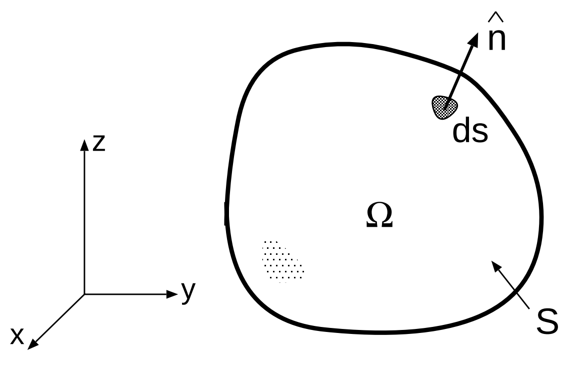

Fig 2-1. A region Ω of fluid bounded by surface S.

Consider a *fixed* region Ω (as shown in Fig. 2-1) of fluid whose density is given by $\rho \equiv \rho(x, y, z, t)$ and velocity by $\vec{u} \equiv \vec{u}(x, y, z, t) \equiv (u, v, w)$. Let S denote the boundary of Ω and $\hat{n}$ the local unit normal outward vector. The instantaneous mass of fluid in the region Ω of the fluid is, in terms of density, $m = \int_\Omega \rho \; d\Omega$. The mass rate of flow out of Ω is given by $\int_S \rho \vec{u} \cdot \hat{n} \; dS$. As mass is neither created nor destroyed, per the principle of conservation of mass, the rate of change of mass in Ω must be equal to the mass rate of flow into Ω: *i.e.*,

$$\frac{d}{dt} \int_\Omega \rho \; d\Omega + \int_S \rho \vec{u} \cdot \hat{n} \; dS = 0 \tag{1}$$

An aside remark: One can obtain this equation (1) directly, by applying the control-volume equation with B corresponding to mass m. As per the definition of a system used earlier in discussing the control-volume equation, (ie. consisting of same fluid particles),

$$\frac{dm_{system}}{dt} = 0 = \int_{\Omega} \rho \; d\Omega + \int_{S} \rho \vec{u} \cdot \hat{n} \; dS$$

Note that the intensive property β corresponding to mass is simply unity, and that in the "flux" integral $\vec{u}_{rel} \cdot \hat{n} = \vec{u} \cdot \hat{n}$ as the control volume is fixed.

Using the Gauss integral identity (with $\vec{A}$ corresponding to $\rho\vec{u}$), we can transform Eqn (1) as

$$\int_{\Omega} \frac{\partial}{\partial t}\rho \; d\Omega + \int_{\Omega} \nabla \cdot (\rho\vec{u}) \; d\Omega = \int_{\Omega} [\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho\vec{u})] d\Omega = 0.$$

The above integral is known as the **integral form of conservation of mass**. As the integral vanishes irrespective of the size or geometry of Ω, and if the density and velocity fields are continuous, then one can show that the integrand must itself be zero: *i.e.*,

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho\vec{u}) = 0,$$

which is called the *differential form of the principle of conservation of mass* or *the equation of continuity*. By expanding the $\nabla \cdot (\rho\vec{u})$ and rearranging terms, we can cast the above equation in a more familiar form:

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho\vec{u}) = \frac{\partial \rho}{\partial t} + [\vec{u}\nabla\rho + \rho\nabla \cdot \vec{u}] = [\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla\rho] + \rho\nabla \cdot \vec{u} = 0.$$

Recognizing that $\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla\rho$ is nothing but the material derivative (total time derivative) of density $\frac{D\rho}{Dt}$, we can rewrite the above equation as

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{u} = 0. \qquad (2)$$

Incompressible Fluid: A fluid is said to be *incompressibe* if $\frac{D\rho}{Dt} = 0$; ie., density of fluid particle is constant. For an incompressible-fluid flow, the equation of continuity given above reduces to

$$\nabla \cdot \vec{u} = 0 \qquad (3)$$

In other words, in an incompressible-fluid flow the velocity field is *divergence-free.*

It is also important to point out that an incompressible fluid need not be homogeneous. Homogeneity implies only that fluid density is same everywhere in the region occupied by the fluid; ie.,

$\nabla\rho = 0$ A fluid that is not homegeneous is called an *inhomogeneous* (obviously!) or in oceanography as a *stratified* fluid. With above definitions of *incompressibility* and *homegeneity*, it is not difficult to imagine a fluid which is stratified and yet incompressible. For example, in the ocean the fluid density is not the same at depths compared to that near the surface, and yet, in the study of oceanographic currents and waves one can assume the ocean to be incompressible in that each fluid particle retains the same density as it moves. Those who have taken Underwater Acoustics I may recall that flows in which the flow speed is very small compared to the speed of sound may assumed as an incompressible fluid flow.

II-2. Balance of Linear Momentum in Inviscid Fluid: Eulers Equation

Next, we shall derive the equation of motion for inviscid fluid based on the Newton's second law of motion. *We shall derive the equation of motion using an inertial frame of reference first.* Also, temporarily for convenience sake, we ignore the effect of viscosity in deriving the equation of motion. We shall extend the derivations to an rotating frame of reference and including viscosity later. As done in the derivation of the equation of continuity, let us consider an arbitrary region Ω of fluid bounded by surface S. The external forces acting on the fluid in Ω are (i) due to pressure, p and (ii) due to gravity g acting in the negative z direction (Fig. 2-2). Lets denote the velocity field, as observed in the **inertial** frame oxyz by $\vec{u}$.

The external force $\vec{F}$, due to gravity and pressure, is given by

$$\vec{F} = -\hat{k} \int_\Omega \rho g \; d\Omega - \int_S p\hat{n} \; dS.$$

Note that pressure is a surface stress and that it acts in the direction opposite of $\hat{n}$ (and hence the negative sign for the surface integral of pressure). Using the control-volume equation (**??**) with B corresponding to linear momentum $m\vec{u}$, one can express the rate of change of linear momentum of a system of fluid which occupies Ω at the instant of time t as:

$$\frac{d}{dt}(m\vec{u})_{sys} \equiv \int_\Omega \frac{\partial \rho\vec{u}}{\partial t} \; d\Omega + \int_S (\rho\vec{u})\vec{u}\cdot\hat{n} \; dS.$$

As per the Newton's second law of motion, the rate of change of linear momentum of a system is equal to the sum of the external forces acting on the system, we have

$$\int_\Omega \frac{\partial \rho\vec{u}}{\partial t} \; d\Omega + \int_S (\rho\vec{u})\vec{u}\cdot\hat{n} \; dS = -\hat{k} \int_\Omega \rho g \; d\Omega - \int_S p\hat{n} \; dS.$$

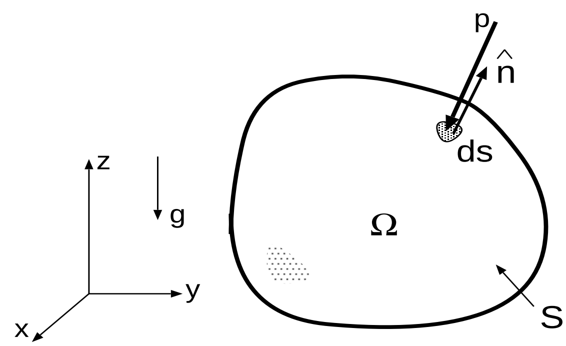

Fig 2-2. Force of pressure and gravity on fluid in region Ω bounded by surface S.

Using Gauss integral identities (see Lecture #1) the surface integrals in the above equation can be transformed to volume integrals:

$$\int_\Omega \frac{\partial \rho \vec{u}}{\partial t}\, d\Omega + \int_\Omega \nabla \cdot [(\rho \vec{u})\vec{u}]\, d\Omega = -\hat{k} \int_\Omega \rho g\, d\Omega - \int_\Omega \nabla p\, d\Omega.$$

which is known as the *integral form of the principle of the balance of momentum of an inviscid fluid.*

> <u>Exercise</u>: Derive integral forms of balance of linear momentum of an inviscid fluid by considering (i) a material volume $\Omega(t)$ which holds the same fluid particles at all times and (ii) by considering a moving region $\Omega(t)$ but not necessarily holding same fluid particles; ie. $\vec{u} \cdot \hat{n} \neq V_n$. How are these different from the one obtained above?

As the above integral relation is true for any Ω in the fluid, and assuming that the flow variables are continuous, we can simply equate the integrals:

$$\frac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot [(\rho \vec{u})\vec{u}] = -\rho g \hat{k} - \nabla p \tag{4}$$

By chain rule,

$$\frac{\partial \rho \vec{u}}{\partial t} \equiv \rho \frac{\partial \vec{u}}{\partial t} + \vec{u} \frac{\partial \rho}{\partial t}$$

and

$$\nabla \cdot [(\rho \vec{u})\vec{u}] \equiv (\rho \vec{u} \cdot \nabla)\vec{u} + \vec{u} \nabla \cdot (\rho \vec{u}),$$

we can write Eqn.(4) as

$$\rho \frac{\partial \vec{u}}{\partial t} + \vec{u} \frac{\partial \rho}{\partial t} + (\rho \vec{u} \cdot \nabla)\vec{u} + \vec{u} \nabla \cdot (\rho \vec{u}) = -\rho g \hat{k} - \nabla p.$$

Grouping first and third terms on the left-hand side together and second and fourth terms together, we can rewrite the above equation as

$$\vec{u}[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u})] + \rho[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}] = -\rho g \hat{k} - \nabla p. \tag{5}$$

By conservation of mass

$$[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u})] = \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0.$$

Therefore, Eqn.(5) becomes

$$\rho[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}] = -\rho g \hat{k} - \nabla p \tag{6}$$

which is known as the **Euler's equation**.

Other Forms of Euler's Equations

In literature, one might find the Euler's equations presented in various forms. We list some of these below.

Recognizing that $\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \equiv \frac{D\vec{u}}{Dt}$, we can write the Euler's equation as:

$$\rho \frac{D\vec{u}}{Dt} = -\rho g \hat{k} - \nabla p. \tag{7}$$

By a vector identity,

$$(\vec{u} \cdot \nabla)\vec{u} \equiv \frac{1}{2}\nabla|\vec{u}|^2 - \vec{u} \times \nabla \times \vec{u}.$$

Therefore the Euler's equation (6) can be written as

$$\rho(\frac{\partial \vec{u}}{\partial t} + \frac{1}{2}\nabla|\vec{u}|^2 - \vec{u} \times \nabla \times \vec{u}) = -\rho g \hat{k} - \nabla p. \tag{8}$$

Note that $\nabla \times \vec{u} \equiv \vec{\omega}$ is called fluid **vorticity**. In terms of $\vec{\omega}$, the Euler's equation above can be expressed as

$$\rho(\frac{\partial \vec{u}}{\partial t} + \frac{1}{2}\nabla|\vec{u}|^2 - \vec{u} \times \vec{\omega}) = -\rho g \hat{k} - \nabla p. \tag{9}$$

In the case of homogeneous-density fluid, the gravity term can be written as

$$-\rho g \hat{k} = -\nabla \rho g z.$$

Thus the right hand side of the Euler's equation can be combined as

$$-\rho g\hat{k} - \nabla p = -\nabla \rho g z - \nabla p = -\nabla(p + \rho g z)$$

For reasons that would become clear later, the quantity $p + \rho g z$ is called the **dynamic pressure**.

To summarize, the Euler's equation for an incompressible, homogeneous fluid can be written in the following forms:

$$\rho[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}] = -\nabla(p + \rho g z). \tag{10}$$

$$\rho \frac{D\vec{u}}{Dt} = -\nabla(p + \rho g z). \tag{11}$$

$$\rho(\frac{\partial \vec{u}}{\partial t} + \frac{1}{2}\nabla|\vec{u}|^2 - \vec{u} \times \vec{\omega}) = -\nabla(p + \rho g z). \tag{12}$$
