Boundary Conditions for Equations Governing Incompressible Fluid Flow

Next, let us examine the boundary conditions for the equations of fluid motion discussed in the previous class. The boundary conditions for an **inviscid-fluid** flow are as follows. On a rigid surface, the velocity must satisfy the no-flux condition; ie.

$$\vec{u} \cdot \hat{n} = V_n$$

where V_n denotes the normal velocity of the surface.

On a free surface, which say is denoted by $z = \eta$, the no-flux condition implies

$$\frac{D(z-\eta)}{Dt} = 0$$

In other words, a free surface is a material surface. The stress vector must be continuous across a free surface; this means that, in the absence of surface tension and viscous stress,

$$p = p_{atm} \text{ on} z = \eta.$$

where p_{atm} denotes the atmospheric pressure. In terms of gage pressure, the above dynamic condition can be written as

$$p = 0 \text{ on} z = \eta.$$

In the particular case of a **potential flow**, the above inviscid-fluid boundary conditions reduce to the following. The no-flux condition on a rigid surface becomes

$$\frac{\partial \phi}{\partial n} = V_n$$

where V_n denotes the normal velocity of the surface. The free-surface kinematic condition remains essentially the same:

$$\frac{D(z-\eta)}{Dt} = 0$$

Substitution of the dynamic condition of $p = 0$ on the free surface in the Euler's integral gives

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0 \quad \text{on } z = \eta$$

Boundary Conditions (VISCOUS Fluid Flow):

In the case of viscous flow, fluid velocity must satisfy both the no-slip and no-flux conditions on a rigid surface. Therefore, on a body that is moving with a velocity $\vec{V}$

$$\vec{u} = \vec{V} \text{ on the body surface } S.$$

The form of free-surface kinematic condition remains the same, at least in the modeling of one-layer flow[1]; ie.,

$$\frac{D(z-\eta)}{Dt} = 0$$

The dynamic condition corresponds to the continuity of the stress vector $\vec{\tau}$ across the free surface. In the absence of atmospheric forcing, the atmospheric pressure on the free surface is zero (gage) and atmospheric viscous stress is also zero. Therefore, the stress vector $\vec{\tau}$ must be equal to zero on the free surface. In a two-dimensional flow, this means

$$\vec{\tau} \cdot \hat{n} = 0, \qquad t\vec{au} \cdot \hat{t} = 0 \text{ on the free surface}$$

where $(\hat{n}, \hat{t})$ denote unit normal and tangential vectors on the surface, respectively. With the following relations, already discussed,

$$\vec{\tau} = \tilde{\sigma} \cdot \hat{n}$$

and

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i} \right)$$

the above conditions can be expanded as

$$\begin{aligned} \vec{\tau} \cdot \hat{n} &= (\tilde{\sigma} \cdot \hat{n}) \cdot \hat{n} \\ &= \left[-p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i} \right) \right] n_i n_j \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{\tau} \cdot \hat{t} &= (\tilde{\sigma} \cdot \hat{n}) \cdot \hat{t} \\ &= \left[-p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i} \right) \right] n_i t_j \\ &= 0 \end{aligned}$$

where the repeated indices imply summation.

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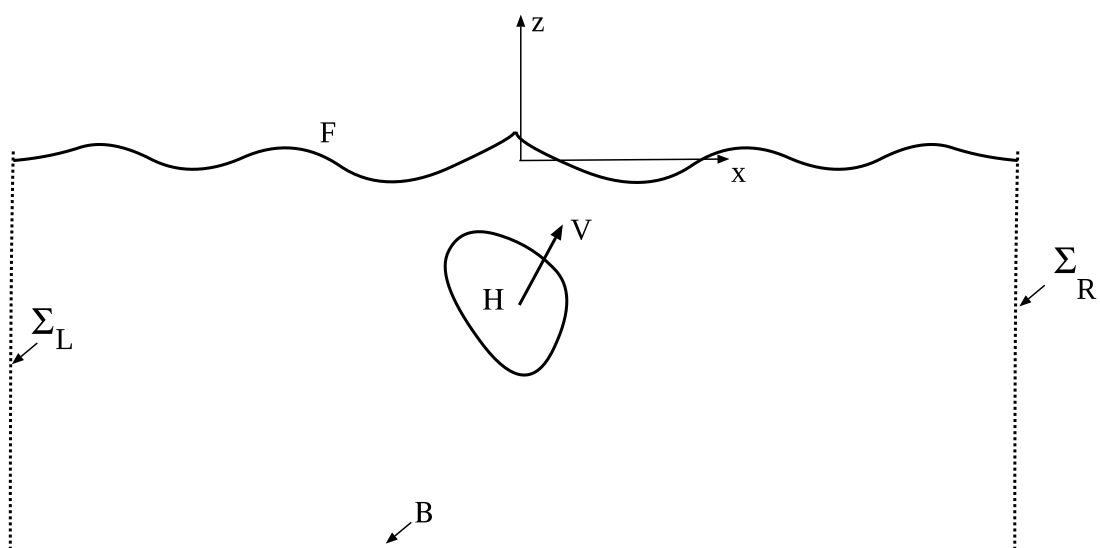
[1] In a two-layer flow modeling, the velocity vector must be continuous across the interface (see Wehausen, *Surface Waves*)

Open Boundary Conditions (Inviscid Potential Flow:

To numerically solve flows in infinite domain, one has to truncate the physical domain using artificial boundaries known as open boundaries. As these are not actual physical boundaries, one has to develop boundary conditions that will make them appear as open. In other words, the open boundary conditions, if properly constructed, will not cause any reflection and will transmit the flow without any modification across the open boundary.

Let us consider a body motion in a free surface causing radiation of waves as illustrated in the figure below. The lateral open boundaries are denoted by Σ_L and Σ_R and the bottom boundary by B.ZZ



In the far field, per Sommerfeld condition, waves must be propagating outward. One can therefore use the advection equation (refer to Lecture Notes 1) to model Σ_L and Σ_R as

$$\frac{\partial \phi}{\partial t} + C\frac{\partial \phi}{\partial x} = 0, \text{ on } \Sigma_R$$

$$\frac{\partial \phi}{\partial t} - C\frac{\partial \phi}{\partial x} = 0, \text{ on } \Sigma_L$$

Here C denotes the wave speed and *phi* the velocity potential. Above equations imply that waves are propagating to the right (outward) at Σ_R and to the left (outward) at Σ_L. In the case of three-dimensional wave motion problem, the correspondng open boundary condition can be written as

$$\frac{\partial \phi}{\partial t} + C\frac{\partial \phi}{\partial R} = 0, \text{ on } \Sigma$$

where R denotes the radial direction from the body towards the far-field open boundary. The above method of modeling the open boundary for transient wave motion problems is referred to as the Orlanski condition in the literature on CFD.

In the case of linear time-harmonic wave radiation problem, in which the unknown potential can be decomposed as

$$\phi(x, y, z, t) = \Phi(x, y, z)\ e^{-i\sigma t}$$

where $i \equiv \sqrt{-1}$ and σ wave (or body motion) frequency, the advection equation can be reduced to the followin equations. In the case of 2D problem, on Σ_R

$$\begin{aligned} & \frac{\partial \phi}{\partial t} + C\frac{\partial \phi}{\partial R} = 0 \\ \rightarrow\ & \left(-i\sigma\Phi + C\frac{\partial \Phi}{\partial x}\right)\ e^{-i\sigma t} = 0 \\ \rightarrow\ & \frac{\partial \Phi}{\partial x} - ik\Phi = 0 \text{ since } C = \sigma/k \text{ where } k\text{denotes the wave number} \end{aligned}$$

Similarly, one can obtain for Σ_L open bondary

$$\frac{\partial \Phi}{\partial x} + ik\Phi = 0$$

In the case of three-dimensional time-harmonic wave problem, the corresponding open-boundary condition is given by

$$\frac{\partial \Phi}{\partial R} - ik\Phi = 0$$

where R denotes the radial outward direction.

By taking the bottom open boundary B to be at a depth that is larger than one-half the wave length (so that the domain can be a deep-water domain), one can set either ϕ or $\frac{\partial \phi}{\partial n}$ to be zero on B.
