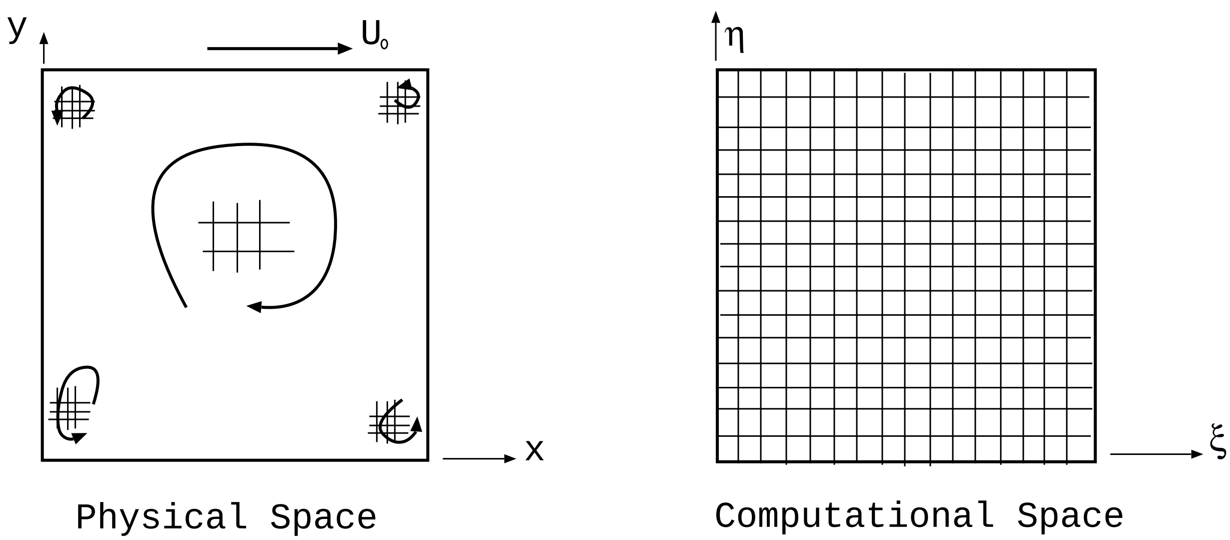
Boundary-Fitted Coordinates - Transformation of Flow Equations

Let us now consider flow equations transformed to the computational space. For illustration let us consider the classical lid-driven cavity flow problem as shown in the figure.



As can be seen, the grid spacing is not uniform; it is coarse in the middle and fine at the corners, presumably so to resolve vortices of all scales well. Let the grids to be time dependent, even though the flow domain remains constant; imagine that grids are generated adpatively as the flow structures evolve.

Governing Equations in the Physical Space (x,y,t): The incompressible Navier-Stokes equations governing the above flow are:

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial \rho}{\partial x} + \nu\nabla^2_{xy}u \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial \rho}{\partial y} + \nu\nabla^2_{xy}v
\end{aligned}
$$

and the unknowns are velocity components u and v and pressure p. In the above equations, ρ denotes density and $\nu \equiv \mu/\rho$ the coefficient of kinematic viscosity with μ being the coeffient of dynamic viscosity. If acceleration of gravity were acting along the negative y direction, p will denote the dynamic pressure.

Transformation Governing Equations to the Computational Space (ξ, η, τ): Using the transformation relations for derivatives obtained earlier (cf. Lecture Notes #15) one can transform the above equations to an uniform computational space. For compactness, let us use subscript notation to denote derivatives: for example, $x_\xi \equiv \frac{\partial x}{\partial \xi}$, $u_\tau \equiv \frac{\partial u}{\partial \tau}$ etc.

<u>Equation of Continuity</u>:

$$y_\eta u_\xi - y_\xi u_\eta - x_\eta v_\xi + x_\xi v_\eta = 0$$

<u>x-component of the momentum equation</u>:

$$u_\tau + \frac{(u - x_\tau)}{J}\left(y_\eta u_\xi - y_\xi u_\eta\right) + \frac{(v - y_\tau)}{J}\left(-x_\eta u_\xi + x_\xi u_\eta\right) = -\frac{1}{\rho J}\left(y_\eta p_\xi - y_\xi p_\eta\right) + \nu\left(\nabla^2_{xy}\xi\ u_\xi + \nabla^2_{xy}\eta\ u_\eta + A u_{\xi\xi} - 2B u_{\xi\eta} + C u_{\eta\eta}\right)$$

$\Rightarrow$

$$u_\tau + \left[\frac{y_\eta(u - x_\tau)}{J} - \frac{x_\eta(v - y_\tau)}{J}\right] u_\xi + \left[\frac{x_\xi(v - y_\tau)}{J} - \frac{y_\xi(u - x_\tau)}{J}\right] u_\eta = \frac{1}{\rho J}\left(y_\eta p_\xi - y_\xi p_\eta\right) + \nu\left[\nabla^2_{xy}\xi\ u_\xi + \nabla^2_{xy}\eta\ u_\eta + A u_{\xi\xi} - 2B u_{\xi\eta} + C u_{\eta\eta}\right]$$

<u>y-component of the momentum equation</u>:

$$v_\tau + \frac{(u - x_\tau)}{J}\left(y_\eta v_\xi - y_\xi v_\eta\right) + \frac{(v - y_\tau)}{J}\left(-x_\eta v_\xi + x_\xi v_\eta\right) = -\frac{1}{\rho J}\left(-x_\eta p_\xi + x_\xi p_\eta\right) + \nu\left(\nabla^2_{xy}\xi\ v_\xi + \nabla^2_{xy}\eta\ v_\eta + A v_{\xi\xi} - 2B v_{\xi\eta} + C v_{\eta\eta}\right)$$

$\Rightarrow$

$$u_\tau + \left[\frac{y_\eta(u - x_\tau)}{J} - \frac{x_\eta(v - y_\tau)}{J}\right] v_\xi + \left[\frac{x_\xi(v - y_\tau)}{J} - \frac{y_\xi(u - x_\tau)}{J}\right] v_\eta = \frac{1}{\rho J}\left(-x_\eta p_\xi + x_\xi p_\eta\right) + \nu\left[\nabla^2_{xy}\xi\ v_\xi + \nabla^2_{xy}\eta\ v_\eta + A v_{\xi\xi} - 2B v_{\xi\eta} + C v_{\eta\eta}\right]$$

In the above,

$$A \equiv \frac{x_\eta^2 + y_\eta^2}{J^2};\ B \equiv \frac{x_\xi x_\eta + y_\xi y_\eta}{J^2};\ C \equiv \frac{x_\xi^2 + y_\xi^2}{J^2};$$

The equations get lengthier upon transformation, a small price worth paying to be able to tackle arbitrary boundary and to be able to resolve flow gradients efficiently and yet be able to solve the problem on an uniform and rectangular mesh in the computational space. In the following chapters we shall discuss various algorithms for the solution of the incompressible Navier-Stokes equations.
