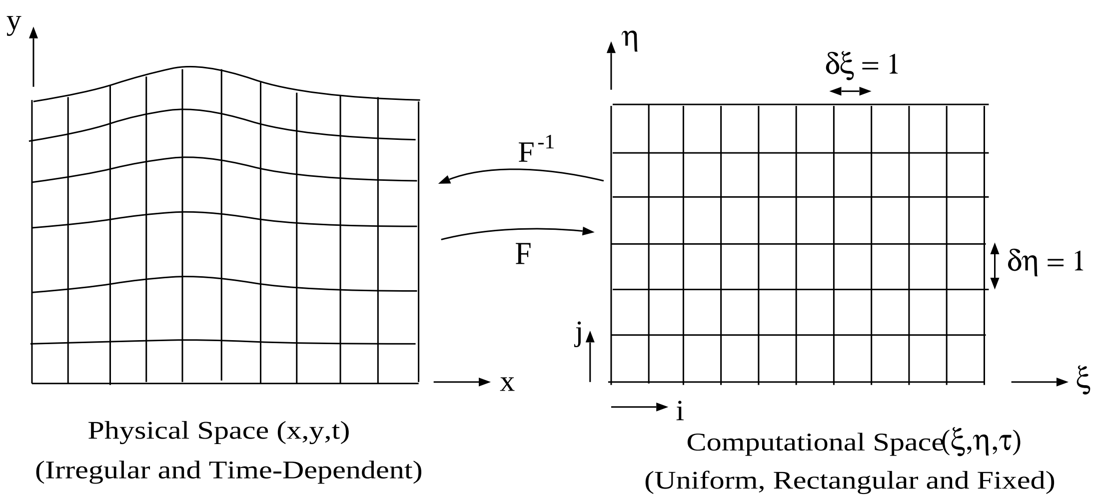
EOC6189 Computational Fluid Dynamics Fall 2006

Finite-Difference Methods Based on Boundary-Fitted Coordinates



In marine hydrodynamics, the physical domain of the problem is often arbitrary and time dependent. To implement a finite-difference method on an uniform and rectangular mesh, the method of boundary-fitted coordinates is used. In this method, the physical space (x, y, t) is mapped to a uniform and rectangular computational space (ξ, η, τ) where times $t = \tau$. The mapping from the physical space to computational space, denoted as F in the figure, is one-to-one and therefore the inverse mapping F^{-1} also exist. The generation of boundary-fitted coordinates in the physical space is essentially determination of the inverse mapping F^{-1} corresponding to uniform mesh of the computational space. We shall study various formulations and equations developed for F^{-1} in the next section.

The governing flow equations formulated originally in the physical space is also transformed to the computational space, using chain-rules of differentiation the transformed equations solved by finite-difference method in the computational space. A few points need to be kept in mind while deriving the transformation relation for derivatives:

1. time t is same everywhere in the computational space; ie., $t_\xi = t_\eta = 0$, where subscript notation is used to denote partial derivative.

2. time τ is same everywhere in the physical space; ie., $\tau_x = \tau_y = 0$.

3. the computational space is fixed; ie., $\xi_t = \eta_t = 0$; and

4. in the case of time-dependent physical domain in which case the grids also change with respect to time, x_τ and y_τ are not equal to zero; these terms represent the rate of change of position and are referred to as "grid speed".

Transformation Relations for Derivatives

By chain-rule of differentiation, one can obtain the following relation for first-order derivatives:

$$\partial_\xi = \partial_x.x_\xi + \partial_y.y_\xi + \partial_t.t_\xi = \partial_x.x_\xi + \partial_y.y_\xi, \quad \text{because } t_\xi = 0$$

$$\partial_\eta = \partial_x.x_\eta + \partial_y.y_\eta + \partial_t.t_\eta = \partial_x.x_\eta + \partial_y.y_\eta, \quad \text{because } t_\eta = 0$$

$$\partial_\tau = \partial_x.x_\tau + \partial_y.y_\tau + \partial_t.t_\tau = \partial_x.x_\tau + \partial_y.y_\tau + \partial_t, \quad \text{because } t_\tau = 1$$

Inverting the above relations, one can show that

$$\begin{aligned}
\partial_x &= \frac{1}{J}\left(y_\eta.\partial_\xi - y_\xi.\partial_\eta\right) \\
\partial_y &= \frac{1}{J}\left(-x_\eta.\partial_\xi + x_\xi.\partial_\eta\right) \\
\partial_t &= \partial_\tau - x_\tau.\frac{1}{J}\left(y_\eta.\partial_\xi - y_\xi.\partial_\eta\right) - y_\tau.\frac{1}{J}\left(-x_\eta.\partial_\xi + x_\xi.\partial_\eta\right)
\end{aligned}$$

Using above relations, one can transform equations involving first-order derivatives to the computational space.

Next, let us consider transformation relation for second-order spatial derivatives, in particular of ∂_{xx} and ∂_{yy} which constitute the Laplacian operator.

$$\begin{aligned}
\partial_x &= \partial_\xi \xi_x + \partial_\eta \eta_x + \partial_\tau \tau_x \\
&= \partial_\xi \xi_x + \partial_\eta \eta_x, \quad \text{because } \tau_x = 0
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\partial_{xx} &= \xi_{xx}\partial_\xi + \xi_x \partial_x(\partial_\xi) + \eta_{xx}\partial_\eta + \eta_x\partial_x(\partial_\eta) \\
&= \xi_{xx}\partial_\xi + \xi_x^2\partial_{\xi\xi} + +\xi_x\eta_x\partial_{\xi\eta} + +\eta_{xx}\partial_\eta + \xi_x\eta_x\partial_{\xi\eta} + \eta_x^2\partial_{\eta\eta} \\
&= \xi_{xx}\partial_\xi + \eta_{xx}\partial_\eta + \xi_x^2\partial_{\xi\xi} + +\eta_x^2\partial_{\eta\eta} + 2\xi_x\eta_x\partial_{\xi\eta}
\end{aligned}$$

Using transformation relations for first-order derivatives ∂_x and ∂_y obtained above,

$$\partial_{xx} = \xi_{xx}\partial_\xi + \eta_{xx}\partial_\eta + \frac{y_\eta^2}{J^2}\partial_{\xi\xi} + +\frac{y_\xi^2}{J^2}\partial_{\eta\eta} - 2\frac{y_\xi y_\eta}{J^2}\partial_{\xi\eta}$$

One can similarly show for ∂_{yy}:

$$\partial_{yy} = \xi_{yy}\partial_\xi + \eta_{yy}\partial_\eta + \frac{x_\eta^2}{J^2}\partial_{\xi\xi} + +\frac{x_\xi^2}{J^2}\partial_{\eta\eta} - 2\frac{x_\xi x_\eta}{J^2}\partial_{\xi\eta}$$

The Laplacian operator in the (x, y, t) space thus transforms as

$$\nabla_{xy}^2 \equiv \partial_{xx} + \partial_{yy} = \nabla_{xy}^2\xi.\partial_\xi + \nabla_{xy}^2\eta.\partial_\eta + \frac{x_\eta^2 + y_\eta^2}{J^2}\partial_{\xi\xi} + \frac{x_\xi^2 + y_\xi^2}{J^2}\partial_{\eta\eta} - 2\frac{x_\xi x_\eta + y_\xi y_\eta}{J^2}\partial_{\xi\eta}$$

One can similarly obtain transformation relations for other derivatives that may be involved in the governing flow equations using chain-rule of differentiation. The fluid dynamics equations formulated in the physical space are transformed to the computational space using the transformation relations.
