EOC6189 **Computational Fluid Dynamics** **Fall 2006**

FINITE DIFFERENCE SOLUTION OF DIFFUSION EQUATION

Next, let us examine, numerical solution of the diffusion equation given by

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \quad \text{and where} \;\; \mu > 0.$$

Let us consider a straigtforward, explicit, forward-time central-space (FTCS) scheme

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \mu \frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\delta x^2}$$

which gives the following recursive relation to advance the solution in time:

$$u_i^{n+1} = u_i^n + \mathcal{K}(u_{i+1}^n + u_{i-1}^n - 2u_i^n), \quad \text{where} \quad \mathcal{K} \equiv \frac{\mu \delta t}{\delta x^2}$$

Consistency and Accuracy. Using Taylor-series expansion, one can easily show that the above scheme is first order accurate in time and second order accurate in space; ie., truncation error = $O(\delta t) + O(\delta x^2)$. As δt and $\delta x \to 0$, one can show that the difference equation reduce to the given differential equation. The schems is therefore consistent.

Stability. Using the von Neumann stability analysis, one can determine the evolution of the amplitude of the numerical solution. Per von Neumann stability analysis, let

$$u_i^n = U^n \; e^{jki\delta x}, \quad \text{where } j = \sqrt{-1}$$

Then, the above scheme gives

$$U^{n+1} = U^n \left(1 + \mathcal{K}[e^{jk\delta x} + e^{-jk\delta x} - 2]\right)$$

or

$$\begin{aligned} \lambda \equiv \frac{U^{n+1}}{U^n} &= 1 + \mathcal{K}(e^{jk\delta x} + e^{-jk\delta x} - 2) \\ &= 1 + 2\mathcal{K}(\cos k\delta x - 1) \\ &= 1 - 2\mathcal{K}(1 - \cos k\delta x) \end{aligned}$$

From above, one can observe that for stability, ie, $|\lambda| \leq 1$,

$$\mathcal{K} \equiv \frac{\mu \delta t}{\delta x^2} \leq \frac{1}{2}$$

The FTCS scheme for the diffusion equation is therfore conditionally stable, the condition being

$$\frac{\mu \delta t}{\delta x^2} \leq \frac{1}{2}$$

Crank-Nicholson Scheme

Let us consider an implicit scheme, given by

$$\frac{u_i^{n+1}-u_i^n}{\delta t} = \frac{\mu}{2}\left(\frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n + u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}}{\delta x^2}\right)$$

To determine the accuracy of the scheme, let us carry out Taylor-series expansion of the terms about node i and discrete time n. For convenience, let us deal with left-hand side and right-hand side separately as follows.

$$\begin{aligned}
\textbf{LHS} &= \frac{u_i^{n+1}-u_i^n}{\delta t} \\
&= \frac{1}{\delta t}\left(u + \frac{\delta t}{1!}u_t + \frac{\delta t^2}{2!}u_{tt} + \frac{\delta t^3}{3!}u_{ttt} + \ldots - u\right) \\
&= u_t + \frac{\delta t}{2!}u_{tt} + \frac{\delta t^2}{3!}u_{ttt} + \ldots
\end{aligned}$$

$$\begin{aligned}
\textbf{RHS} &= \frac{\mu}{2}\left(\frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n + u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}}{\delta x^2}\right) \\
&= \frac{\mu}{2\delta x^2}\left(u + \frac{\delta x}{1!}u_x + \frac{\delta x^2}{2!}u_{xx} + \frac{\delta x^3}{3!}u_{xxx} + .. + u - \frac{\delta x}{1!}u_x + \frac{\delta x^2}{2!}u_{xx} - \frac{\delta x^3}{3!}u_{xxx} + .. - 2u\right) \\
&+ \frac{\mu}{2\delta x^2}\left(+u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2u_i^{n+1}\right) \\
&= \frac{\mu}{2\delta x^2}\left(\delta x^2 u_{xx} + \delta x^2[u_{xx} + \delta t u_{xxt} + \ldots]\right) \\
&= \mu u_{xx} + \frac{\mu}{2\delta x^2}[\delta x^2 \delta t u_{txx} + \ldots] \\
&= \mu u_{xx} + O(\delta x^2) + \frac{\mu}{2}\delta t u_{txx} + \ldots \\
&= \mu u_{xx} + \frac{\delta t}{2}u_{tt} + O(\delta t^2) \quad (\text{since } \mu u_{xx} = u_t\ )
\end{aligned}$$

Cancelling the $\frac{\delta t}{2}u_{tt}$ on the LHS and RHS above, we observe that the scheme solves

$$u_t = \mu u_{xx} + O(\delta x^2) + O(\delta t^2)$$

The scheme is thus second-order accurate in time and space.

Note that the scheme is implicit and involves values only at two instants of time n and $n+1$. The accuracy in time is also of second order.

Stability: Assignment Examine the stability of the Crank-Nicholson scheme for the diffusion equation using von Neumann analysis; ie. by studying growth/decay of spectral solution of the form

$$u_i^n = U^n\ e^{jki\delta x} \quad \text{where } j = \sqrt{-1}$$