EOC6189 Computational Fluid Dynamics Fall 2006

VON NEUMANN STABILITY ANALYSIS

In this lecture, we discuss a general method to determine the stability of a finite-difference scheme. The method is based on spectral (discrete Fourier transform) analysis. To explain the method, let us consider the solution of the advection equation

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0, \quad \text{where } c = \text{ constant and } u \equiv u(x,t), \quad \text{and } u(x,0) = f(x)$$

using the first order forward-time backward-space scheme

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + c\frac{u_i^n - u_{i-1}^n}{\delta x} = 0$$

which can be rewritten as

$$u_i^{n+1} = u_i^n + \mathcal{C}\left(u_{i-1}^n - u_i^n\right)$$

where $\mathcal{C}$ denotes the Courant number, defined as

$$\mathcal{C} \equiv \frac{c\ \delta t}{\delta x}$$

Now, let us examine the time evolution of wave-like solution of the form

$$u_i^n = U^n\ e^{jki\delta x}, \quad \text{where } j \equiv \sqrt{-1}$$

where U^n denotes the amplitude of the solution at discrete time n. Substituting in the above scheme, we obtain

$$e^{jki\delta x}\left(U^{n+1} = U^n\left[1 + \mathcal{C}[e^{-jk\delta x} - 1]\right]\right)$$

Or,

$$\frac{U^{n+1}}{U^n} = 1 + \mathcal{C}[e^{-jk\delta x} - 1]$$

The ratio of solution amplitudes at subsequent times is called the amplification factor λ; ie.,

$$\lambda \equiv \frac{U^{n+1}}{U^n}$$

The scheme is said to be stable if $|\lambda| \leq 1$ and unstable if $|\lambda| > 1$. Note that $|\lambda| = 1$ corresponds to neutral stability. In the above scheme,

$$|\lambda| = |1 + \mathcal{C}[e^{-jk\delta x} - 1]| = |1 - \mathcal{C}[1 - e^{-jk\delta x}]| \leq 1, \quad \text{if} \quad \mathcal{C} \leq 1$$

In other words, the scheme is stable provided the Courant number $\mathcal{C} \leq 1$. The FTBS (upwind) scheme is therefore said to be conditionally stable with the condition being $\mathcal{C} \leq 1$.

Carrying out the similar analysis for the FTFS (downwind) scheme

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + c\frac{u_{i+1}^n - u_i^n}{\delta x} = 0$$

for the solution of

$$u_t + cu_x = 0, \quad \text{where } u \equiv u(x,t) \;\; \text{and } u(x,0) = f(x)$$

one can show that the amplification factor

$$\lambda \equiv \frac{U^{n+1}}{U^n} = 1 + \mathcal{C}[1 - e^{jk\delta x}]$$

or

$$|\lambda| = |1 + \mathcal{C}[1 - e^{jk\delta x}]| > 1, \quad \text{since } \; \mathcal{C} > 0$$

In other words, the amplification factor is always greater than 1 meaning that the scheme is unconditionally unstable. Note that we had arrived at similar finding based on the coefficient of numerical viscosity, earlier.

Finally, let us consider an implicit upwind scheme given by

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + c\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\delta x} = 0$$

Examining the evolution of wave-like solution of the form

$$u_i^n = U^n \; e^{jki\delta x}$$

we obtain for the above scheme, the following relation for the amplification factor:

$$U^{n+1}\left(1 + \mathcal{C}[1 - e^{-jk\delta x}]\right) = U^n$$

$\rightarrow$

$$\lambda \equiv \frac{U^{n+1}}{U^n} = \frac{1}{1 + \mathcal{C}[1 - e^{-jk\delta x}]}$$

which is always less than 1 (since $\mathcal{C} > 0$). The scheme is therefore said to be unconditionally stable.

Assignment: Examine the stability of the leap-frog scheme given by

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\delta t} + c\frac{u_{i+1}^n - u_{i-1}^n}{2\delta x} = 0$$

for the solution of

$$u_t + cu_x = 0, \quad \text{where } u \equiv u(x,t) \;\; \text{and } u(x,0) = f(x)$$

(Hint: Noting that $U^{n+1} = \lambda^2 U^{n-1}$, $U^n = \lambda U^{n-1}$, one should obtain a quadratic equation for λ. For stability, both the roots must be less than 1)
