EOC6189 **Computational Fluid Dynamics** **Fall 2006**

UPWIND DIFFERENCE SCHEME FOR THE ADVECTION EQUATION

In the previous lecture, we introduced the notion of consistency and accuracy of a finite-difference scheme and also examined spurious truncation errors such as artificial viscosity and dispersion. For discussion, we used the forward-time forward-space (FTFS) scheme

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + c\frac{u_{i+1}^n - u_i^n}{\delta x} = 0$$

for the solution of

$$u_t + cu_x = 0, \quad \text{where } u \equiv u(x,t) \ \text{ and } u(x,0) = f(x)$$

whose exact solution is $u(x,t) = f(x - ct)$. We found that the scheme, even though consistent and $O(\delta t) + O(\delta x)$ accurate, it is not a stable scheme owing to negative artificial viscosity for any δt and δx values. We also saw that the scheme can introduce dispersion, not present in the original equation, unless $c\ \delta t = \delta x$.

Let us now examine another scheme, forward-time backward-space, given by

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + c\frac{u_i^n - u_{i-1}^n}{\delta x} = 0$$

for the solution of

$$u_t + cu_x = 0, \quad \text{where } u \equiv u(x,t) \ \text{ and } u(x,0) = f(x)$$

The above scheme yields the following time recursive relation

$$u_i^{n+1} = u_i^n + \frac{c\delta t}{\delta x}\left(u_{i-1}^n - u_i^n\right)$$

to advance the solution in time.

Using Taylor series expansion about node i and discrete time n, one can show that the scheme is equivalent to solving

$$u_t + cu_x + \left(\frac{\delta t}{2!}u_{tt} + \frac{\delta t^2}{3!}u_{ttt} + ... - \frac{\delta x}{2!}u_{xx} + \frac{\delta x^2}{3!}u_{xxx} + ...\right) = 0$$

But when $\delta t = 0$ and $\delta x = 0$, the above reduces to the differential equation it is intended to solve; namely,

$$u_t + cu_x = 0$$

Therefore, as the difference equation reduces to the differential equation when $\delta t = 0$ and $\delta x = 0$, the FTBS scheme is a consistent scheme.

Again by Taylor-series expansion the terms in the scheme about i and n, one can obtain

$$\frac{1}{\delta t}\left(u + \frac{\delta t}{1!}u_t + \frac{\delta t^2}{2!}u_{tt} + \frac{\delta t^3}{3!}u_{ttt} + ... - u\right) + \frac{c}{\delta x}\left(u - u + \frac{\delta x}{1!}u_x - \frac{\delta x^2}{2!}u_{xx} + \frac{\delta x^3}{3!}u_{xxx} + ...\right) = 0$$

$\longrightarrow$

$$u_t + cu_x + \left(\frac{\delta t}{2!}u_{tt} + \frac{\delta t^2}{3!}u_{ttt} + ... - \frac{c\delta x}{2!}u_{xx} + \frac{c\delta x^2}{3!}u_{xxx} + ...\right) = 0$$

$\longrightarrow$

$$u_t + cu_x + O(\delta t^1) + O(\delta x^1) = 0$$

The FTBS scheme is thus first-order accurate in both space and time.

Let us spurious truncation error effects of the FTBS scheme. The FTBS scheme,

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + c\frac{u_i^n - u_{i-1}^n}{\delta x} = 0$$

for non-zero δt and δx solves

$$u_t + cu_x + \left(\frac{\delta t}{2!}u_{tt} + \frac{\delta t^2}{3!}u_{ttt} + ... - \frac{c\delta x}{2!}u_{xx} + \frac{c\delta x^2}{3!}u_{xxx} + ...\right) = 0$$

with the truncation error introduced by the terms in the (..) paranthesis. With the leading terms of truncation, the equation that the scheme solves become

$$u_t + cu_x = -\frac{\delta t}{2!}u_{tt} + \frac{c\delta x}{2!}u_{xx}$$

From the given differential equation,

$$u_t = -cu_x \quad \rightarrow u_{tt} = -c(u_t)_x = -c(-cu_x)_x = c^2 u_{xx}$$

The equation, including leading terms of truncation, that the scheme solves is thus

$$\begin{aligned} u_t + cu_x &= -\frac{\delta t}{2!}u_{tt} + \frac{c\delta x}{2!}u_{xx} \\ &= \left(-\frac{\delta t}{2!}c^2 + \frac{c\delta x}{2!}\right)u_{xx} \\ &= \nu_{num}\ u_{xx} \end{aligned}$$

where ν_{num} is the coefficient of artificial or numerical viscosity and for the present scheme, it is given by

$$\nu_{num} = \left(-\frac{\delta t}{2!} c^2 + \frac{c \delta x}{2!} \right)$$

For the coefficient of numerical viscosity ν_{num} to be positive,

$$c\delta t \ < \delta x$$

otherwise, ν_{num} will be negative. From the knowledge of diffusion equation, we know that negative viscosity will lead to exponential growth of the solution in time making the scheme unstable. For numerical viscosity viewpoint, we thus observe, that the FTBS scheme is conditionally stable, the condition being

$$c\delta t \ \leq \delta x$$

Note that $c\delta t = \delta x \rightarrow \nu_{num} = 0$ which means no artificial viscosity will be present in the solution; in this case, the scheme is neutrally (ie. neither growth nor decay) stable.

With next-order of terms in the truncated series retained, one can show that the FTBS solves the following equation

$$u_t + cu_x \quad = \quad \nu_{num} \ u_{xx} - \frac{\delta t^2}{3!} u_{ttt} - \frac{c \delta x^2}{3!} u_{xxx}$$

With $u_t = -cu_x \quad \rightarrow \quad u_{ttt} = -c^3 u_{xxx}$, above becomes

$$u_t + cu_x = \nu_{num} \ u_{xx} + d_{num} \ u_{xxx}$$

where the “coefficient” of numerical dispersion is given by

$$d_{num} = c^3 \frac{\delta t^2}{3!} - \frac{c \delta x^2}{3!}$$

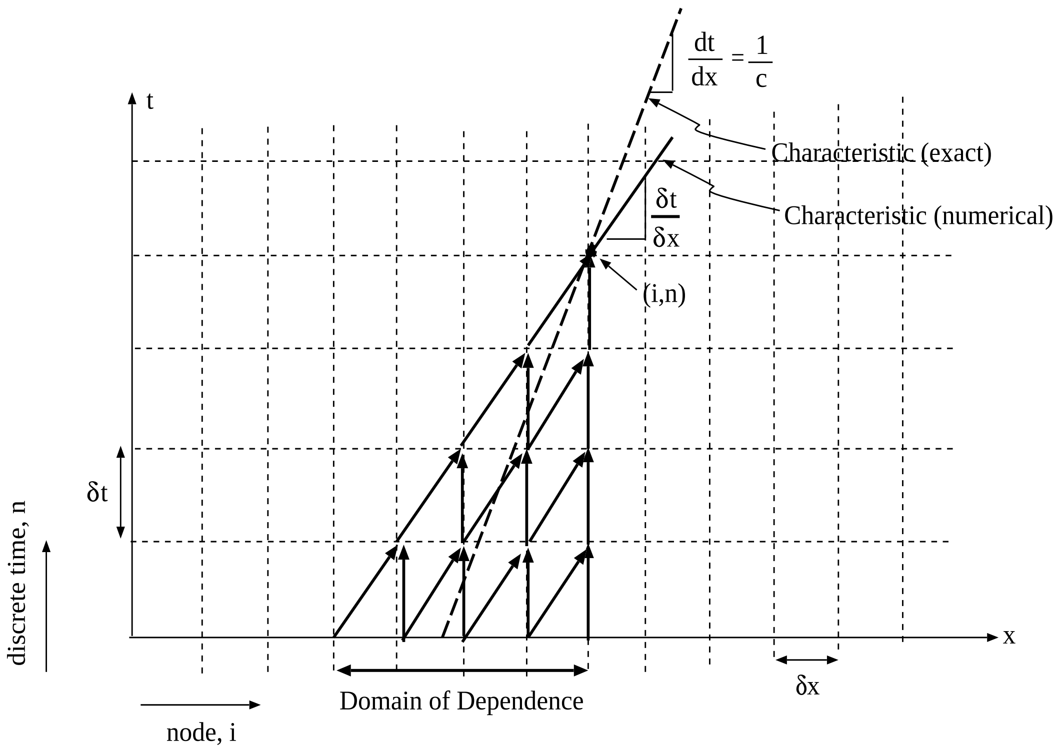
which is the same as in the FTFS scheme! The numerical dispersion will results in different wave-number components of the data $u(x, 0) = f(x)$ travelling at different speeds and thereby leading to spurious dispersion. Interestingly, as seen before, that artificial effect of dispersion can be made zero by choosing δt and δx such that $c\delta t = \delta x$:

$$d_{num} = c^3 \frac{\delta t^2}{3!} - \frac{c \delta x^2}{3!} = 0, \quad \text{if} \quad c\delta t = \delta x$$

Thus by having δt and δx satisfy $c\delta t = \delta x$, one can get rid off both the artificial viscosity and artificial dispersion in the FTBS scheme!

Convergence

Let us examine the FTBS scheme graphically, as in the figure below.



The numerical solution at spatial and temporal node (i,n) depends on that at (i,n-1) and (i-1,n-1); that at (i,n-1) and (i-1,n-1) in turn depend on that at (i,n-2); (i-1,n-2) and (i-1,n-2); (i-2,n-2) respectively, and so on. Thus tracing back in time, as shown in the figure, one can see that the numerical solution at (i,n) depends on the intial value at a few nodes at intial time t=0 which constitute the domain of dependence. The slope of the hypotenuse of the corresponding region is $\delta t/\delta x$. The exact solution propagates along the characteristic with slope $dt/dx = 1/c$. For the numerical solution to have some semblance of the exact solution, the characterisitic passing through (i,n) should originate from within the domain of dependence. That means, $dt/dx = 1/c \ \geq \ \delta t/\delta x$ (or) $c\delta t/\delta x \ \leq 1$. The numerical solution will converge to the exact solution, as δt and $\delta x \rightarrow 0$ satisfying $c\delta t/\delta x \ \leq 1$. Note that convergence of a scheme relates to numerical and exact solutions, as opposed to consistency relating difference and differential equations.