FINITE-DIFFERENCE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

In this lecture, we consider finite-difference solution of some simple partial differential equations, exact solutions of which are known as discussed earlier in the course. Let us begin numerical solution of partial-differential equations with the advection equation given by

$$u_t + cu_x = 0, \quad \text{where } u \equiv u(x,t) \text{ and } u(x,0) = f(x)$$

Note that we have used subscript-notation to denote derivatives. The exact solution of the above is given by

$$u(x,t) = f(x - ct)$$

which corresponds to intial data advecting in the positive x direction with the constant speed c .

Generally speaking, per finite-difference method, the derivatives are approximated by difference schemes and the resulting algebraic system solved directly or iteratively. For the present advection equation, let us consider the following scheme

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + c\frac{u_{i+1}^n - u_i^n}{\delta x} = 0$$

in which both the time-derivative and the spatial-derivative terms are approximated using first-order forward-difference scheme. The above scheme is called forward-time forward-space scheme, for obvious reason, and also as a "downwind"scheme, the reason for which will be given later. Note that we have used subscript i to denote spatial nodes along the x axis and superscript n to denote discrete time. In other words, $x = i\ \delta x$ and $t = n\ \delta t$ where δx and δt denote mesh width and time-step size, respectively. The above schemes, when re-writen, gives the following recursive relation to advance the solution in time:

$$u_i^{n+1} = u_i^n + \frac{c\delta t}{\delta x}\left(u_i^n - u_{i+1}^n\right)$$

CONSISTENCY

A scheme is said to be consistent, if the difference equation reduces to the differential equation as δx and $\delta t \rightarrow 0$. One can check the consistency of a difference scheme using Taylor series. Let us examine now the consisency of the forward-time forward-space scheme given above. Taylor-series expanding the terms about i and n, we obtain

$$\frac{1}{\delta t}\left(u + \frac{\delta t}{1!}u_t + \frac{\delta t^2}{2!}u_{tt} + \frac{\delta t^3}{3!}u_{ttt} + ... - u\right) + \frac{c}{\delta x}\left(u + \frac{\delta c}{1!}u_x + \frac{\delta x^2}{2!}u_{xx} + \frac{\delta x^3}{3!}u_{xxx} + ... - u\right) = 0$$

$\rightarrow$

$$u_t + cu_x + \left(\frac{\delta t}{2!}u_{tt} + \frac{\delta t^2}{3!}u_{ttt} + ... + \frac{\delta x}{2!}u_{xx} + \frac{\delta x^2}{3!}u_{xxx} + ...\right) = 0$$

As $\delta x,\ \delta t\ \rightarrow 0$, the above scheme becomes

$$u_t + cu_x = 0$$

which is the given partial differential equation. The scheme is therefore a consistent scheme.

ACCURACY

Accuracy of a scheme refers to the order of magnitude of the truncation resulting in the finite-difference approximation of the differential equation. A scheme is said be $O(\delta t^n) + O(\delta x^m)$ accurate, or n-th order scheme in time and m-th order in space, if the order of magnitude of the truncated terms is $O(\delta t^n)$ + $O(\delta x^m)$. One can again use Taylor series to determine a scheme's accuraccy. Let us examine the accuracy of the forward-time and forward-space scheme under consideration. Taylor-series expanding, about $x = i\delta x$ and $t = n\delta t$,

$$\frac{1}{\delta t}\left(u + \frac{deltat}{1!}u_t + \frac{\delta t^2}{2!}u_{tt} + \frac{\delta t^3}{3!}u_{ttt} + ... - u\right) + \frac{c}{\delta x}\left(u + \frac{deltax}{1!}u_x + \frac{\delta x^2}{2!}u_{xx} + \frac{\delta x^3}{3!}u_{xxx} + ... - u\right) = 0$$

$\rightarrow$

$$u_t + cu_x + \left(\frac{\delta t}{2!}u_{tt} + \frac{\delta t^2}{3!}u_{ttt} + ... + \frac{c\delta x}{2!}u_{xx} + \frac{c\delta x^2}{3!}u_{xxx} + ...\right) = 0$$

$\rightarrow$

$$u_t + cu_x + O(\delta t^1) + O(\delta x^1) = 0$$

The scheme is thus first-order accurate in both space and time. Note that for a scheme to be consistent, it must be at the least first-order accurate.

SPURIOUS EFFECTS OF TRUNCATION - (i) Artificial Viscosity

Note that in a finite-difference scheme, δt and δx are small but not zero; therefore, the numerical solution will exhibit properties of the truncated terms as well which must not be taken to be real effects. Let us examine the effect of the leading terms of the truncated series in the forward-time forward-space scheme that we have considered earlier to discuss consistency and accuracy. The scheme given by

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + c\frac{u_{i+1}^n - u_i^n}{\delta x} = 0$$

for non-zero δt and δx solves

$$u_t + cu_x + \left(\frac{\delta t}{2!}u_{tt} + \frac{\delta t^2}{3!}u_{ttt} + ... + \frac{c\delta x}{2!}u_{xx} + \frac{c\delta x^2}{3!}u_{xxx} + ... \right) = 0$$

with the truncation error introduced by the terms in the (..) paranthesis. With the leading terms of truncation, the equation that the scheme solves become

$$u_t + cu_x = -\frac{\delta t}{2!}u_{tt} - \frac{c\delta x}{2!}u_{xx}$$

From the given differential equation[1]:

$$u_t = -cu_x \quad \rightarrow u_{tt} = -c(u_t)_x = -c(-cu_x)_x = c^2 u_{xx}$$

Therefore, the equation the scheme solves is

$$\begin{aligned} u_t + cu_x &= -\frac{\delta t}{2!}u_{tt} - \frac{c\delta x}{2!}u_{xx} \\ &= -\left(\frac{\delta t}{2!}c^2 + \frac{c\delta x}{2!} \right) u_{xx} \\ &= \nu_{num}\; u_{xx} \end{aligned}$$

where ν_{num} is the coefficient of artificial or numerical viscosity and for the present scheme, it is given by

$$\nu_{num} = -\left(\frac{\delta t}{2!}c^2 + \frac{c\delta x}{2!} \right)$$

[1]Perhaps we could use a different letter say v to denote exact solution; but we will stick to the same letter u to denote both numerical and exact solutions, which is which will be obvious from the context.

which is negative for any δt and δx. Recall from earlier discussion on diffusion equation

$$u_t = \nu u_{xx}$$

that negative ν means exponential growth of the solution in time. Thus the forward-time forward-space scheme under consideration, which has negative artificial viscosity, will lead to instability.

SPURIOUS EFFECTS OF TRUNCATION - (ii) Artificial Dispersion

Because of truncation errors, the numerical solution can also show artificial dispersive effects (ie. different wave numbers propagating at different speeds) even though in the original advection equation $u_t + cu_x = 0$ all wave numbers travel at the constant speed c. With the next order of terms retained in the forward-time forward-space scheme, under discusstion, the scheme solves

$$u_t + cu_x \quad = \quad \nu_{num}\ u_{xx} - \frac{\delta t^2}{3!} u_{ttt} - \frac{c\delta x^2}{3!} u_{xxx}$$

With $u_t = -cu_x \quad \rightarrow \quad u_{ttt} = -c^3 u_{xxx}$, above becomes

$$u_t + cu_x = \nu_{num}\ u_{xx} + d_{num}\ u_{xxx}$$

where the "coefficient" of numerical dispersion is given by

$$d_{num} = c^3 \frac{\delta t^2}{3!} - \frac{c\delta x^2}{3!}$$

As discussed earlier in the course, the equation $u_t = du_{xxx}$ will result in different wave numbers traveling at different speeds as

$$c = d.k^2, \quad \text{ie., } c\ \alpha\ k^2$$

ie, shorter waves will travel faster. Thus because of artificial dispersion in the forward-time forward-space scheme, the initial data $f(x)$ will lose its shape in due course of time because of dispersion with shorter waves heading outward fast. Note, interestingly, that artificial effect of dispersion can be made zero by choosing δt and δx such that $c\delta t = \delta x$:

$$d_{num} =^3 \frac{\delta t^2}{3!} - \frac{c\delta x^2}{3!} = 0, \quad \text{if} \quad c\delta t = \delta x$$

. To continue.