

FINITE-DIFFERENCE METHOD

Next, let us examine field-discretization methods for solving flow equations. In the finite-difference method, solutions are sought at discrete points in space, known as nodes, and discrete instants of time. The governing equations are discretized and the resulting equations solved by direct or iterative methods.

Finite-Difference Approximation of Derivatives

In the finite-difference method, derivatives are approximated by difference schemes. Some schemes, which may be already familiar to you, for the first derivative of a continuous function are

$$\begin{aligned} \frac{df}{dx} &\approx \frac{f_{i+1} - f_i}{\delta x} \quad (\text{forward difference scheme}) \\ &\approx \frac{f_i - f_{i-1}}{\delta x} \quad (\text{backward difference scheme}) \\ &\approx \frac{f_{i+1} - f_{i-1}}{2\delta x} \quad (\text{central difference scheme}) \end{aligned}$$

where the subscript i denotes the node and δx the distance between the nodes; in other words, $f_i \equiv f(x = i.\delta x)$.

Accuracy of a Difference Scheme

Using Taylor-series expansion, which is a convergent series, one can establish the accuracy of a finite-difference scheme. As an illustrative example, let us consider the central-difference scheme

$$\frac{f_{i+1} - f_{i-1}}{2\delta x} \quad \text{used for the approximation of } \frac{df}{dx} \text{ at } x=i \delta x$$

Taylor-series expanding the f_{i+1} and f_{i-1} terms in the scheme about $x = i.\delta x$, we obtain

$$\begin{aligned} \frac{f_{i+1} - f_{i-1}}{2\delta x} &= \frac{1}{2\delta x} \left[f + \frac{\delta x}{1!} f_x + \frac{\delta x^2}{2!} f_{xx} + \frac{\delta x^3}{3!} f_{xxx} \dots \right] \\ &\quad - \frac{1}{2\delta x} \left[f - \frac{\delta x}{1!} f_x + \frac{\delta x^2}{2!} f_{xx} - \frac{\delta x^3}{3!} f_{xxx} \dots \right] \\ &= f_x + \frac{\delta x^2}{3!} f_{xxx} + \dots \\ &= f_x + O(\delta x^2) \end{aligned}$$

As the truncated series is of the order of $O(\delta x^2)$, the above central-difference scheme is a second-order accurate scheme. In the above determination of accuracy, for brevity, subscripts are used to

denote derivatives; for example,

$$f_{xx} \equiv \frac{d^2 f}{dx^2}$$

One can similarly establish, that the forward- and backward-difference schemes given on the previous page, are first-order accurate schemes. It is worthwhile to point out that the central-difference scheme is second-order accurate **only** if δx is constant. If the distance between nodes i and $i + 1$, call it δ_{x+} , is not that same as the distance between the nodes i and $i - 1$, call it δ_{x-} , then the scheme will only be first-order accurate! You can check this fact for yourself by examining the Taylor-series expansion of f_{i+1} and f_{i-1} about i

A General Method to Determine Finite-Difference Scheme

One could use Taylor-series expansion, for developing a difference scheme for the approximation of a derivative of a continuous function. To illustrate the method, let us consider, as an example, the approximation of df/dx at node i , using f at nodes i , $i + 1$, and $i + 2$. As first step, express function values at the nodes involved in terms of function and its derivatives at the node where we are seeking the approximation of the derivative; in other words, in the present case, Taylor-series expand f at nodes i , $i + 1$ and $i + 2$ about the node i .

$$\begin{aligned} f_i &= f_i \\ f_{i+1} &= f_i + \frac{\delta x}{1!} f_x + \frac{\delta x^2}{2!} f_{xx} + \frac{\delta x^3}{3!} f_{xxx} + \frac{\delta x^4}{4!} + \dots \\ f_{i+2} &= f_i + \frac{2\delta x}{1!} f_x + \frac{4\delta x^2}{2!} f_{xx} + \frac{8\delta x^3}{3!} f_{xxx} + \frac{16\delta x^4}{4!} + \dots \end{aligned}$$

Note that the distance between the nodes is denoted by δx , a constant, and the subscript notation is used to denote derivatives. Linearly combining the above, we obtain

$$af_i + bf_{i+1} + cf_{i+2} = (a+b+c) \cdot f_i + \frac{\delta x}{1!} f_x (b+2c) + \frac{\delta x^2}{2!} f_{xx} (b+4c) + \frac{\delta x^3}{3!} f_{xxx} (b+8c) + \frac{\delta x^4}{4!} f_{xxxx} (b+16c) + \dots$$

where a, b and c are unknown constants. As our objective is approximation for the derivative f_x , we set

$$a + b + c = 0, \quad b + 2c = 1, \quad b + 4c = 0$$

solution of which yields

$$a = -3/2, \quad b = +2, \quad \text{and} \quad c = -1/2$$

Note that we can have equations only for three terms on the right-hand as the unknowns are only three. With above values of a , b and c , the above linear combination becomes

$$\frac{-3f_i + 4f_{i+1} - f_{i+2}}{2} = 0 + \frac{\delta x}{1!} f_x \cdot 1 + 0 + \frac{\delta x^3}{3!} f_{xxx} (-2) + \dots$$

Dividing by δx , we obtain

$$\frac{-3f_i + 4f_{i+1} - f_{i+2}}{2\delta x} = 0 + f_x \cdot 1 + 0 + \frac{\delta x^2}{3!} f_{xxx} (-2) + \dots$$

which can be written also as

$$f_x \equiv \frac{df}{dx} = \frac{-3f_i + 4f_{i+1} - f_{i+2}}{2\delta x} + O(\delta x^2)$$

One can observe that the above one-sided forward-difference scheme is second-order accurate.

One can similarly show, using Taylor-series expansion, that an one-sided backward difference scheme for df/dx is given by

$$f_x \equiv \frac{df}{dx} = \frac{+3f_i - 4f_{i-1} + f_{i-2}}{2\delta x} + O(\delta x^2)$$

and it is also second-order accurate.

As one other example, let us determine approximation for d^2f/dx^2 at node i , using f at i , $i + 1$ and $i - 1$. Taylor-series expanding f at i , $i + 1$ and $i - 1$, about the node i :

$$\begin{aligned} f_i &= f_i \\ f_{i+1} &= f_i + \frac{\delta x}{1!} f_x + \frac{\delta x^2}{2!} f_{xx} + \frac{\delta x^3}{3!} f_{xxx} + \frac{\delta x^4}{4!} + \dots \\ f_{i-1} &= f_i - \frac{\delta x}{1!} f_x + \frac{\delta x^2}{2!} f_{xx} - \frac{\delta x^3}{3!} f_{xxx} + \frac{\delta x^4}{4!} + \dots \end{aligned}$$

Linearly combining, one gets

$$af_i + bf_{i+1} + cf_{i-1} = (a+b+c) \cdot f_i + \frac{\delta x}{1!} f_x (b-c) + \frac{\delta x^2}{2!} f_{xx} (b+c) + \frac{\delta x^3}{3!} f_{xxx} (b-c) + \frac{\delta x^4}{3!} f_{xxxx} (b+c) + \dots$$

where a, b and c are unknowns. As our objective is approximation for the second derivative f_{xx} , we set

$$a + b + c = 0, \quad b - c = 0, \quad b + c = 1$$

solution of which yields $a = -1$, $b = 1/2$ and $c = 1/2$. Thus the linear combination becomes

$$\frac{-2f_i + f_{i+1} + f_{i-1}}{2} = 0 + 0 + \frac{\delta x^2}{2!} f_{xx} 1 + 0 + \frac{\delta x^4}{3!} f_{xxxx} 1 + \dots$$

Dividing by $\delta x^2/2!$,

$$\frac{-2f_i + f_{i+1} + f_{i-1}}{\delta x^2} = 0 + 0 + f_{xx} 1 + 0 + \frac{\delta 2! x^2}{3!} f_{xxxx} 1 + \dots$$

or,

$$f_{xx} \equiv \frac{d^2f}{dx^2} = \frac{-2f_i + f_{i+1} + f_{i-1}}{\delta x^2} + O(\delta x^2)$$

which is a second-order accurate central-difference scheme.