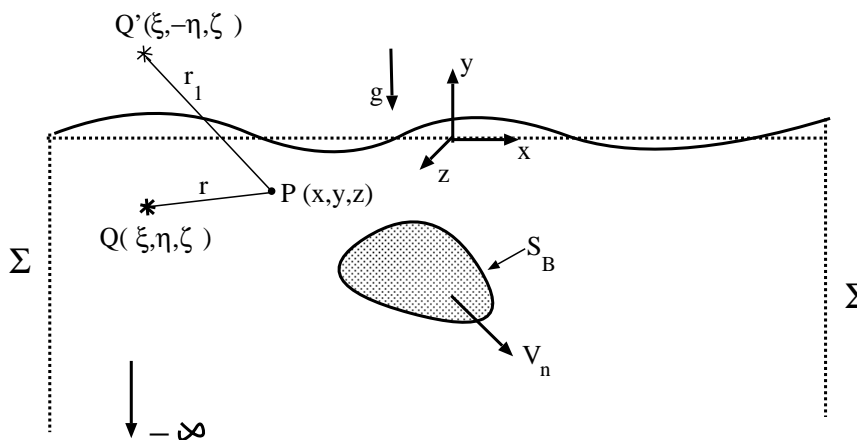


FREQUENCY-DOMAIN 3D FREE-SURFACE GREEN'S FUNCTION

Use of the Green's function corresponding to a simple source in infinite fluid ($\ln r$ in 2D, $1/r$ in 3D) for the solution of linear wave-body interaction problems will involve all the boundaries in the Green's theorem, which could be computationally demanding memory-wise. Use of simple source Green's function together with Lagrangian treatment of free-surface conditions, on the other hand, allows one to tackle fully-nonlinear wave-body interaction problems as discussed in the previous lecture. For linear problems, it is possible to derive a Green's function that accounts for free-surface and far-field boundary conditions so that the Green's theorem will involve only integrals over the body surface. In the present lecture, we present the formulation and solution of 3D frequency-domain free-surface Green's function.

Consider a body under sinusoidal motion in a free-surface as shown in the figure below.



The body and far-field boundaries are represented by S_B and Σ , respectively. The calm surface corresponds to $y=0$ on which, in the linear wave problems, free-surface conditions are satisfied. Let V_n denote the body normal velocity and σ the frequency of body oscillation. Let us take the fluid to be infinitely-deep.

Governing equations for the velocity potential. In the linearized problem, which will correspond to small-amplitude body and wave motions, the velocity potential will be of the form

$$\Phi(x, y, z, t) = \phi(x, y, z) e^{-i\sigma t}$$

and the governing equations of the potential amplitude $\phi(x, y, z)$ are

$$\nabla^2 \phi = 0 \text{ everywhere in the fluid}$$

$$\frac{\partial \phi}{\partial n} = V_n \text{ on the mean body surface}$$

$$\phi = 0, \text{ as } y \rightarrow -\infty$$

The combined free-surface condition is given by

$$\frac{\partial \phi}{\partial y} - \nu \phi = 0, \text{ on } y = 0$$

where $\nu \equiv \sigma^2/g$ denotes the frequency parameter.

In the lateral far-field, the flow will satisfy the Sommerfeld radiation condition:

$$\left(\frac{\partial}{\partial R} - ik\right)\sqrt{R}\phi = 0$$

where R denotes the horizontal radial distance ($=\sqrt{x^2 + z^2}$) and k the wave number which in deep water is equal to σ^2/g . Note that a factor \sqrt{R} is introduced to account for the decay of potential (and wave) amplitude in the radial direction, per conservation of energy. Time variation of the

potential being taken to be of the form $e^{-i\sigma t}$, the Sommerfeld condition in other words states that the far-field behavior of ϕ to be of the form

$$\frac{1}{\sqrt{R}} e^{ikR}$$

so that the phase function is $(kR - \sigma t)$ which represents waves propagating radially outwards.

Governing equations for the Green's function. The Green's function in this problem corresponds to the potential of a pulsating source in the free surface, pulsating in time with frequency σ . The amplitude of the potential, denoted by G , satisfies

$$\nabla^2 G = \delta(r - 0)$$

where r denotes the distance from the source location. Above means that the Laplacian of G is zero everywhere except at the source where it is singular. As in the case of ϕ ,

$$G = 0, \text{ as } y \rightarrow -\infty$$

and in the far field (in the radial direction)

$$\left(\frac{\partial}{\partial R} - ik\right)\sqrt{R}G = 0$$

The combined free-surface condition for G is given by

$$\frac{\partial G}{\partial y} - \nu G = 0, \text{ on } y = 0$$

where, as before,

$$\nu \equiv \frac{\sigma^2}{g}$$

Solution for the Green's function. The above equations can be solved as follows. For reasons that will become apparent in due course, let us decompose G as

$$G = \frac{1}{r} + \frac{1}{r_1} + H$$

where r denotes the distance from the source point $Q(\xi, \eta, \zeta)$ to the field point $P(x, y, z)$ and r_1 the distance from the field point P to the image of Q about the calm free surface denoted as $Q'(\xi, -\eta, \zeta)$ (see figure on the previous page). In other words,

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$$

$$r_1 = \sqrt{(x - \xi)^2 + (y + \eta)^2 + (z - \zeta)^2}$$

Note that the singularity is now only in the $1/r$ term (which corresponds to the potential due to a source in infinite fluid). The term $1/r_1$ is not singular for the image is not in the fluid domain. The term H which corresponds to wave boundary conditions is regular everywhere. With above decomposition for G , the equations for H become

$$\nabla^2 H = 0$$

since

$$\nabla^2 \frac{1}{r} = \delta(r - 0); \quad \nabla^2 \frac{1}{r_1} = 0$$

$$H = 0 \text{ as } y \rightarrow -\infty$$

since

$$\frac{1}{r} \text{ and } \frac{1}{r_1} = 0 \text{ as } y \rightarrow -\infty$$

In the far-field Σ :

$$\left(\frac{\partial}{\partial R} - ik\right)\sqrt{R}H = 0 \text{ as } R \rightarrow \infty$$

as

$$\frac{1}{r} \text{ and } \frac{1}{r_1} = 0 \text{ as } R \rightarrow \infty$$

The combined free-surface condition yields the following inhomogeneous condition for H :

$$\frac{\partial \phi}{\partial y} - \nu G = 0, \text{ on } y = 0$$

→

$$\frac{\partial H}{\partial y} - \nu H = -\frac{\partial}{\partial y} \frac{1}{r} - \frac{\partial}{\partial y} \frac{1}{r_1} + \nu \frac{1}{r} + \nu \frac{1}{r_1}, \text{ on } y = 0$$

One can show, that with $y=0$, above reduces to

$$\frac{\partial H}{\partial y} - \nu H = \frac{2\nu}{\sqrt{(x - \xi)^2 + \eta^2 + (z - \zeta)^2}}$$

Solution of H

One can solve the above set of equations governing the wave-part of the Green's function H using Fourier Transform. Let the Fourier transformation of H from (x', z') space to the corresponding $(k_{x'}, k_{z'})$ space be written as

$$H^{**} = \frac{1}{2\pi} \int dx' \int dz' H e^{-i(k_{x'}x' + k_{z'}z')}$$

and

$$H = \frac{1}{2\pi} \int dk_{x'} \int dk_{z'} H^{**} e^{+i(k_{x'}x' + k_{z'}z')}$$

where

$$x' \equiv (x - \xi)$$

and

$$z' \equiv (z - \zeta)$$

The horizontal distance between the source and field points can therefore be written as

$$R \equiv \sqrt{x'^2 + z'^2}$$

or, equivalently, in terms of R,

$$x' = R \cos\alpha$$

$$z' = R \sin\alpha$$

where

$$\alpha = \tan^{-1} \frac{z'}{x'}$$

In view of the wave-number components $k_{x'}$ and $k_{z'}$, a wave-number vector can be defined as

$$\vec{k} = k_{x'} \hat{i} + k_{z'} \hat{k}, \quad \text{and } k \equiv |\vec{k}| = \sqrt{k_{x'}^2 + k_{z'}^2}$$

In terms of k and phase

$$\theta = \tan^{-1} \frac{k_{z'}}{k_{x'}}$$

$$k_{x'} = k \cos\theta, \quad k_{z'} = k \sin\theta$$

The above Fourier can thensformation refore be also written as

$$\begin{aligned} H^{**} &= \frac{1}{2\pi} \int dx' \int dz' H e^{-i(k_{x'}x' + k_{z'}z')} \\ &= \frac{1}{2\pi} \int dx' \int dz' H e^{-i(k \cos\theta x' + k \sin\theta z')} \\ &= \frac{1}{2\pi} \int dR \int R d\alpha H e^{-i(k \cos\theta \cdot R \cos\alpha + k \sin\theta \cdot R \sin\alpha)} \\ &= \frac{1}{2\pi} \int dR \int R d\alpha H e^{-ikR \cos(\theta - \alpha)} \end{aligned}$$

And the inverse transform as

$$\begin{aligned} H &= \frac{1}{2\pi} \int dk_{x'} \int dk_{z'} H^{**} e^{+i(k_{x'}x' + k_{z'}z')} \\ &= \frac{1}{2\pi} \int dk \int k d\theta H^{**} e^{+i(k \cos\theta x' + k \sin\theta z')} \\ &= \frac{1}{2\pi} \int dk \int k d\theta H^{**} e^{-i(k \cos\theta \cdot R \cos\alpha + k \sin\theta \cdot R \sin\alpha)} \\ &= \frac{1}{2\pi} \int dk \int k d\theta H^{**} e^{+ikR \cos(\theta - \alpha)} \end{aligned}$$

Substituting the Fourier-Transform relation for H in the Laplace equation, we get the following ordinary differential equation for H^{**} :

$$(-ik)^2 H^{**} + \frac{d^2}{dy^2} H^{**} = 0$$

solution of which is

$$H^{**} = C e^{ky} + D e^{-ky}$$

As $H \rightarrow 0$ as $y \rightarrow -\infty$,

$$H^{**} = A(k) e^{|k|y}$$

where $A(k)$ is the integration constant. Using the transform in the free-surface condition

$$\frac{\partial H}{\partial y} - \nu H = \frac{2\nu}{\sqrt{(x - \xi)^2 + \eta^2 + (z - \zeta)^2}}$$

we obtain

$$(|k| - \nu) H^{**}|_{y=0} = f^{**}$$

where

$$f \equiv \frac{2\nu}{\sqrt{(x - \xi)^2 + \eta^2 + (z - \zeta)^2}}$$

Therefore,

$$A(k) = \frac{f^{**}}{|k| - \nu}$$

and thus

$$H^{**} = \frac{f^{**}}{|k| - \nu} e^{|k|y}$$

Having so determined H^{**} one find H as inverse transform of H^* .

Using the integral identities given in the Appendix (at the end of this Lecture), one can show that

$$\begin{aligned} f^{**} &= \left(\frac{2\nu}{r} \Big|_{y=0} \right)^{**} \\ &= \frac{2\nu}{k} e^{-k|\eta|} \end{aligned}$$

and therefore

$$H^{**} = \frac{2\nu}{k(|k| - \nu)} e^{|k|y} e^{-k|\eta|}$$

Taking the inverse transform, and noting the relations in the Appendix, one can obtain

$$\begin{aligned} H &= \frac{1}{2\pi} \int_0^\infty dk k \int_{-\pi}^\pi d\theta e^{ikR \cos(\theta-\alpha)} H^{**} \\ &= \frac{1}{2\pi} \int_0^\infty dk k \int_{-\pi}^\pi d\theta e^{ikR \cos(\theta-\alpha)} \frac{2\nu}{k(|k| - \nu)} e^{|k|y} e^{-k|\eta|} \\ &= \frac{1}{2\pi} \int_0^\infty dk k \int_{-\pi}^\pi d\theta \frac{2\nu}{k(|k| - \nu)} e^{k(y+\eta)} e^{ikR \cos(\theta-\alpha)} \\ &= 2\nu \int_0^\infty dk \frac{e^{k(y+\eta)}}{k - \nu} J_o(kR) \end{aligned}$$

The 3D frequency-domain free-surface Green's function is therefore

$$\begin{aligned} G &= \frac{1}{r} + \frac{1}{r_1} + H \\ &= \frac{1}{r} + \frac{1}{r_1} \nu \int_0^\infty dk \frac{e^{k(y+\eta)}}{k - \nu} J_o(kR) \end{aligned}$$

Evaluation of the integral requires caution in that the integrand is singular at $k = \nu \equiv \sigma^2/g$.

The singular contribution of the integral can be determined by carrying out the integral on the complex k plane and noting the pole on the real axis for $k = \nu$. The contribution of the pole can be determined using the residue theorem; the sense of the contour, ie the contour encloses the pole below of above the real k axis, is to be determined using the Sommerfeld radiation condition. The final expression for G is given by

$$\begin{aligned} G &= \frac{1}{r} + \frac{1}{r_1} + \int_0^\infty dk \frac{e^{k(y+\eta)}}{k - \nu} J_o(kR) \text{ (with } k \neq \nu) \\ &+ \pi i 2\nu e^{\nu(y+\eta)} J_o(\nu R) \text{ (which is the contribution of the integral for } k = \nu) \end{aligned}$$

Reference to articles on the subject, including numerical evaluation of the integral, will be given later.

Appendix

[A1]

$$\frac{1}{(x^2 + y^2 + z^2)^{1/2}} = \frac{1}{2\pi} \int_0^\infty dk k \int_{-\pi}^{+\pi} d\theta \frac{e^{-k|y|}}{k} e^{ik(x \cos\theta + z \sin\theta)}$$

[A2]

$$\frac{1}{r_1}, \frac{1}{r} = \frac{1}{[(x - \xi)^2 + (y \pm \eta)^2 + (z - \zeta)^2]^{1/2}} = \frac{1}{2\pi} \int_0^\infty dk k \int_{-\pi}^{+\pi} d\theta \frac{e^{-k|y \pm \eta|}}{k} e^{ik(x' \cos\theta + z' \sin\theta)}$$

where $x' = x - \xi$, and $z' = z - \zeta$.

[A3] With $x' = R \cos\alpha$ and $z' = R \sin\alpha$

$$\frac{1}{r_1}, \frac{1}{r} = \frac{1}{[(x - \xi)^2 + (y \pm \eta)^2 + (z - \zeta)^2]^{1/2}} = \frac{1}{2\pi} \int_0^\infty dk k \int_{-\pi}^{+\pi} d\theta \frac{e^{-k|y \pm \eta|}}{k} e^{ikR \cos(\theta - \alpha)}$$

[A4] Bessel Function of the First Kind and Zeroth Order

$$J_0(kR) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta e^{ikR \cos(\theta - \alpha)}$$

[A5]

$$\frac{1}{r_1}, \frac{1}{r} = \int_0^\infty dk e^{-k|y \pm \eta|} J_0(kR)$$