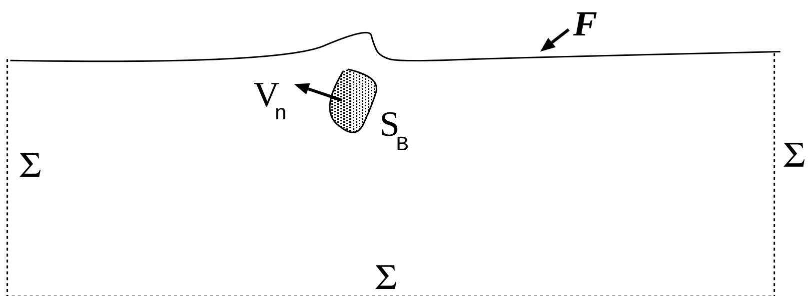
Mixed Eulerian-Lagrangian Solution of Full-Nonlinear Wave Motion Problem

Here, we present a boundary-integral method based on mixed Eulerian-Lagrangian formulation, originally developed by Longuet-Higgins and Cokelet (1978), to solve fully-nonlinear wave motion problems in the time domain. As name suggests, the method involves an Eulerian description of velocity and pressure fields while Lagrangian treatment of the free-surface conditions. The original method was developed to study plunging-type wave breakers; here we will present the method as applied to a typical wave-body interaction in open domain as ilustrated in the following figure.



The governing equations for the velocity potential are the familiar Laplace equation

$$\nabla^2\phi = 0$$

with velocity given by

$$\vec{u} = \nabla\phi$$

and pressure by the Euler's integral

$$p = -\rho g y - \rho\frac{\partial\phi}{\partial t} - \frac{\rho}{2}|\nabla\phi|^2$$

with y axis taken to be zero on the calm surface and pointing upwards.

On the body, the potential has to satisfy the no-flux condition

$$\frac{\partial \phi}{\partial n} = V_n$$

where V_n is the normal velocity of the body.

On the bottom open boundary, taken to be sufficiently deep, one can set ϕ to be zero.

For the time-domain simulation, with initial condition corresponding to quiescent state:

$$\phi, Y = 0, \quad \text{at} t = 0$$

(where Y denotes the free surface elevation), one can use one of the following assumption to model the open boundary in the lateral direction

$$\phi = 0, \text{ until body generated waves reach } \Sigma$$

For a longer time simulation, allowing waves to pass through Σ, one may also consider the advection equation and determine the velocity potential on Σ:

$$\frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial R} = 0$$

where R denotes radially outward direction on Σ and c the local wave celerity (which of course has to be determined numerically and locally for nonlinear wave motion problems. For example, a scheme of the following type can be used to advance *phi* on Σ in time:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\delta t} + c^n \frac{\phi_i^n - \phi_{i-1} n}{\delta R} 0$$

where the superscripts denote discrete time instants, the subscripts the spatial nodes along the radial direction, c^n numerically-determined local wave speed, δt the time-step size used in the simulation and δR the mesh size. Above requires determination of ϕ on two adjacent shells at Σ to evaluate the c^n and $\frac{\partial \phi}{\partial R}|^n$ on Σ. Details on the numerics, such as stability and accuracy, and implementation techniques will be presented in later chapters. Here, a plausible scheme is suggested only to explain the overall algorithm for nonlinear wave motion analysis in the time domain.

On the free surface, the boundary conditions can be written in Lagrangian form as follows. As the free surface is a material surface, the particles on the surface have to move with particle velocity; in other words,

$$\frac{DX}{Dt} = \frac{\partial \phi}{\partial x}; \quad \frac{DY}{Dt} = \frac{\partial \phi}{\partial y}; \quad \frac{DZ}{Dt} = \frac{\partial \phi}{\partial z}$$

where (X, Y, Z) denote coordinates of free-surface particles and the right-hand side terms the components of fluid velocity on the free-surface. By time integrating the above equations, one can advance the position of free-surface particles in time, and thus track the free surface. A simple scheme that may be considered (even though will be shown later to be not an ideal scheme from stability viewpoint) is

$$\frac{X^{n+1} - X^n}{\delta t} = \frac{\partial \phi}{\partial x}^n; \quad \frac{Y^{n+1} - Y^n}{\delta t} = \frac{\partial \phi}{\partial y}^n; \quad \frac{Z^{n+1} - Z^n}{\delta t} = \frac{\partial \phi}{\partial z}^n$$

which advances free-surface particle coordinates from discrete time n to $n + 1$.

By noting that

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla \phi \cdot; \quad \text{and} \quad \nabla \phi \cdot \nabla \phi = |\nabla \phi|^2,$$

the free-surface dynamic condition given by

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + gY = 0$$

can be written in the Lagrangian form (one involing material time derivative D/Dt) as

$$\frac{D\phi}{Dt} - \frac{1}{2}|\nabla \phi|^2 + gY = 0$$

or as

$$\frac{D\phi}{Dt} = \frac{1}{2}|\nabla \phi|^2 - gY = 0$$

which can be time integrated, for example as

$$\frac{\phi^{n+1} - \phi^n}{\delta t} = \frac{1}{2}|\nabla \phi^n|^2 - gY^{n+1}$$

to determine velocity potential on the free surface at the next time level $n + 1$.

In sum, on the body, the normal derivative of the velocity potential is known from the no-flux condition. On the deep open bottom, ϕ may be set to zero. On the lateral open boundary, ϕ may be set to zero until waves reach the open boundary or solve the advection equation to determine ϕ on the open boundary at each time step. On the free surface, by time integrating (numerically) the Lagrangian forms of kinematic and dynamic conditions, one can advance the free surface in time and also determine the velocity potential on the free surface at each time step.

Now, let us examine the 3D Green's theorem with $P \in S$:

$$2\pi\phi(P) + \int_S \phi(Q) \left(\frac{\partial}{\partial n}\frac{1}{r}\right)_Q dS = \int_S \frac{1}{r}\frac{\partial \phi}{\partial n}_Q dS$$

where S now consists of body surface S_B, open boundaries Σ and the free surfac F. On the open boundary and free surface, the velocity potential is known (either specified or obtained by time integration of equations involving ϕ) and on the body boundary S_B, $\partial\phi/\partial n = V_n$. Keeping the unknown terms to the left and the known term to the right, the Green's theorem can be written as

$$2\pi\phi(\text{if } P \in S_B) + \int_{S_B} \phi(Q) \left(\frac{\partial}{\partial n}\frac{1}{r}\right)_Q dS - \int_{F+\Sigma} \frac{1}{r}\frac{\partial \phi}{\partial n}_Q dS$$
$$= -2\pi\phi(\text{if } P \in F, \Sigma) + \int_{S_B} \frac{1}{r}\frac{\partial \phi}{\partial n}_Q dS - \int_{F+\Sigma} \phi(Q) \left(\frac{\partial}{\partial n}\frac{1}{r}\right)_Q dS$$

Knowing the right-hand side, the above can be solved to determine ϕ on the body and $\frac{\partial\phi}{\partial n}$ on Σ and the free surface F. As seen before, the above integral-equation upon discretization will yield a matrix equation of the type

$$[A]\,(\psi) = (b)$$

where $[A]$ denotes the coefficient matrix, which by the way is time-dependent because of evolving geometries in time, column matrix (ψ) unknowns ϕ on the body and $\partial\phi/\partial n$ on F and Σ and the column matrix (b) the right-hand side integrals. In the index notation, above matrix equation can be written as

$$A_{ij}\psi_j = b_i$$

Above equation has to be solved at each time step subject to updated values of velocity potentials on the free surface (and open boundary) and instantaneous body-normal velocity. The coefficient matrices have to be determined at each time step as they are also time-dependent because of changing geometry.

Upon determining ϕ on the body surface, one can determine $\partial\phi/\partial t$ or $D\phi/Dt$ on the body surface; using it in the Euler's integral, one can find pressure and by integration along the body surface the time history of the hydrodynamic force on the body.

Knowing ϕ (by integrating the free surface dynamic condition) and $\partial\phi/\partial n$ (after solving the Green's theorem) on F, one can determine orthogonal components of fluid velocity on the free surface which will be needed to advance the free-surface position in time.

Above is the essence of the method. Subtle details are not presented at this point, but will be taken up later, after discussing finite-difference method to integrate differential equations.
