Numerical Solution of the Green's Theorem.

Let us examine a numerical method to solve the integral equation, given by the Greens theorem, corresponding to a body motion in infinite fluid. Let us consider the two-dimensional case, as the basic principle of the method is essentially the same for 2D and 3D problems.

For field point P on the body surface, S_B, per Green's theorem

$$-\pi \phi P + \int_{S_B, P \neq Q} \phi(Q) \frac{\partial}{\partial n_Q} \ln r \, dS_Q = \int_{S_B, P \neq Q} \ln r \, \frac{\partial \phi}{\partial n_Q} \, dS_Q, \quad \text{for } P \in S_B$$

where r denotes the distance between the field point $P = (x, y)$ and source point $Q = (\xi, \eta)$:

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

Note that quantities inside the integrals correspond to source-point coordinates (ξ, η) and not field point coordinates (x, y). Therefore, for example, $\frac{\partial}{\partial n_Q} \ln r$ means

$$\begin{aligned} \frac{\partial}{\partial n_Q} \ln r &= \nabla|_{(\xi,\eta)} \ln r \ \cdot n_Q \\ &= \frac{\partial}{\partial \xi} \ln\sqrt{(x-\xi)^2 + (y-\eta)^2} \; n_\xi + \frac{\partial}{\partial \eta} \ln\sqrt{(x-\xi)^2 + (y-\eta)^2} \; n_\eta \\ &= \frac{(\xi - x) n_\xi + (\eta - y) n_\eta}{(x-\xi)^2 + (y-\eta)^2} \end{aligned}$$

where n_ξ and n_η denote the components of the unit normal vector at the source point (ξ, η).

Discretization

Let us consider a very simple discretization of the geometry and the integral equation. Let the body be discretized by straight-line elements as show in the figure on the next page and let the unknown potential ϕ be constant on each element (or panel). Let i denote the field panel and j the source panel. For each i, the integrals involved in the Greens theorem have to be carried out for $j = 1, ...N$ but $i \neq j$. The discretized integral equation, with $P \in S_B$

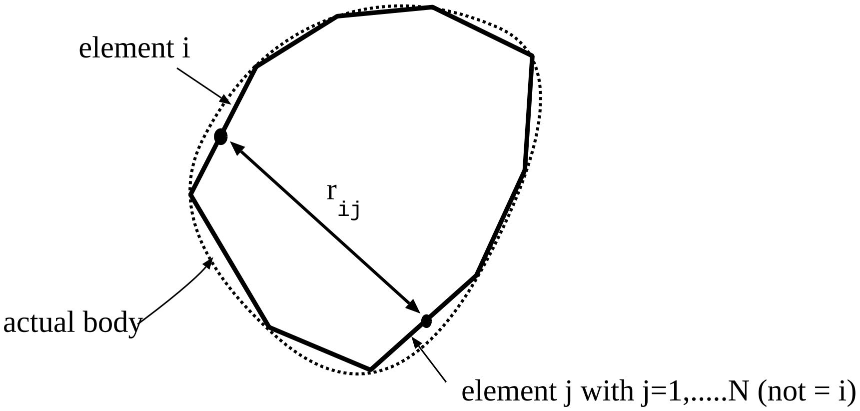
$$-\pi\phi P + \int_{S_B, P\neq Q} \phi(Q)\frac{\partial}{\partial n_Q} \ln r \; dS_Q = \int_{S_B, P\neq Q} \ln r \; \frac{\partial\phi}{\partial n_Q} \; dS_Q, \quad \text{for } P \in S_B$$

where, as just shown,

$$\frac{\partial}{\partial n_Q} \ln r = \frac{(\xi - x)n_\xi + (\eta - y)n_\eta}{(x-\xi)^2 + (y-\eta)^2}$$

can therefore be written as

$$-\pi\phi_i + \Sigma_{j=1,j\neq i}^{N} \frac{(x_j - x_i)n_{x,j} + (y_j - y_i)n_{y,j}}{(x_i - x_j)^2 + (y_i - y_j)^2} \; \delta S_j = \Sigma_{j=1,j\neq i}^{N} \ln\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \; V_{nj} \; \delta S_j$$



In the above numerical version of the Green's theorem,

- (x_i, y_i) denote the coordinates of the central point (controlpoint) of the field panel i,
- (x_j, y_j) denote the coordinates of the central point (controlpoint) of the source panel j,
- N the total number of panels on the body,
- ϕ_i the potential on the field panel i,
- ϕ_j the potential on the source panel j,

- $(n_{x,j}, n_{y,j})$ the unit normal vector components of the source panel j (note that n has point into the body, ie. out of the fluid)
- V_{nj} the normal velocity of the body at the source panel j, and
- δS_j the length of the panel j.

The above equation can be written as a matrix equation

$$[A]\,(\phi) = (b)$$

where the (b) matrix represents the summation of known quantities on the right-hand side of the integral equation. The matrix $[A]$, called the coefficient matrix, correspond to the summation involving "dipole" distributions, on the left-hand side of the integral equation. Above equation can also be written in the "index"notation as

$$A_{ij}\phi_j = b_i$$

The above equation can be solved, for example after inverting the A matrix as

$$(\phi) = [A]^{-1}\,(b)\,, \quad \text{ie.,} \quad \phi_j = A_{ij}^{-1} b_i$$

Upon thus determining ϕ on the body surface, one can, if interested, determine ϕ at any point P inside the fluid domain using the Greens theorem

$$-2\pi\phi P = -\int_{S_B, P\neq Q} \phi(Q)\frac{\partial}{\partial n_Q}\ \mathrm{ln} r\ dS_Q + \int_{S_B, P\neq Q} \mathrm{ln} r\ \frac{\partial \phi}{\partial n_Q}\ dS_Q, \quad \text{for } P \in \Omega$$

as

$$-2\pi\phi_i = -\Sigma_{j=1}^{N}\ \frac{(x_j - x_i)n_{x,j} + (y_j - y_i)n_{y,j}}{(x_i - x_j)^2 + (y_i - y_j)^2}\ \delta S_j + \Sigma_{j=1}^{N}\ \mathrm{ln}\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}\ V_{nj}\ \delta S_j$$

Note that, having solved for ϕ on the body surface, both summations on the right involve known quantities. The summations over the body surface will not encounter $i = j$, as the point i is in the fluid now. In other words (x_i, y_i) here denote the coordinates of the field point P which is now in Ω.