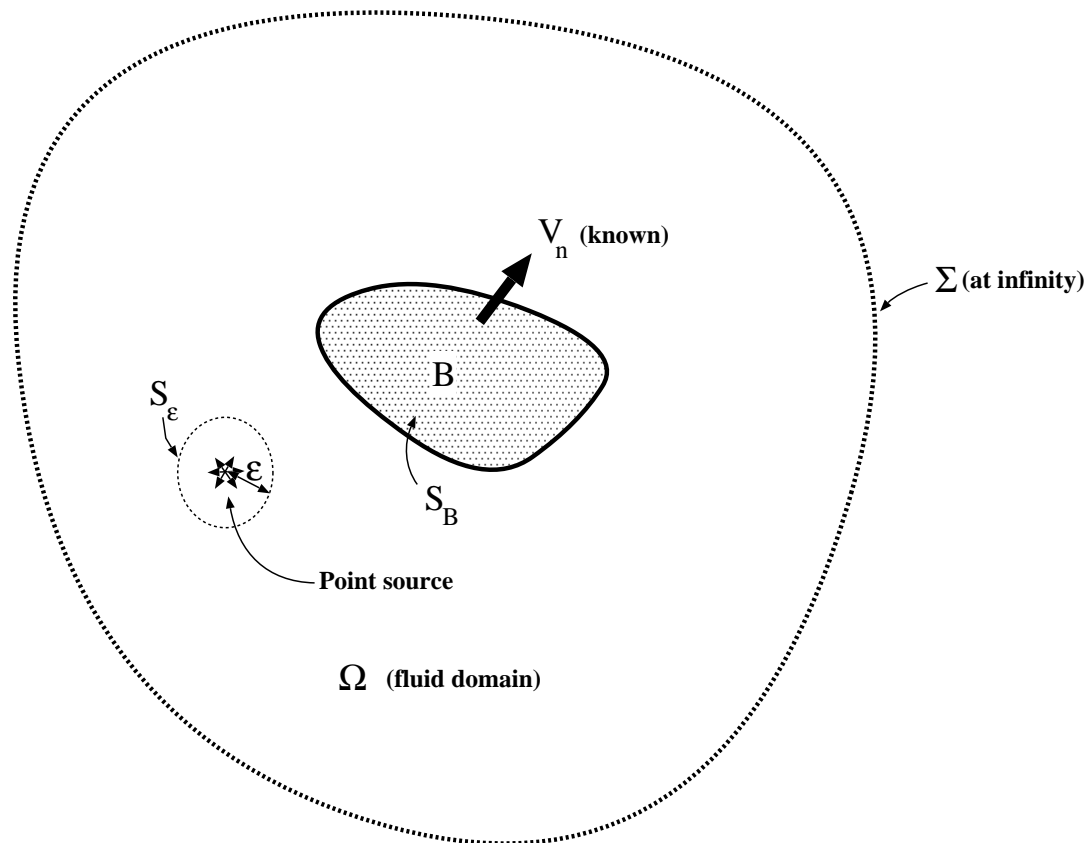


Green's Theorem / Green's Mixed Distribution.

In today's lecture, we continue the development of the boundary-integral formulation for the analysis of potential flow about an arbitrary body in infinite fluid.



The formulation is based on the infinite-fluid Green's function which we derived in an earlier class.

In 2D,

$$G = \frac{1}{2\pi} \ln r$$

and in 3D

$$G = -\frac{1}{4\pi} \frac{1}{r}$$

where r denotes the distance between the source point Q with coordinates (ξ, η, ζ) and field point P with coordinates (x, y, z) ; ie.,

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$$

Gauss Theorem:

Recall, for any continuous vector field defined in a region Ω

$$\int_{\Omega} \nabla \cdot \vec{A} \, d\Omega = \int_S \vec{A} \cdot \hat{n} \, dS$$

where S denotes the boundary of Ω and \hat{n} unit normal outward vector on S. Letting

$$\vec{A} = \phi \nabla G$$

where ϕ denotes the potential due to body motion and G the potential (Greens function) corresponding to a point source, we obtain

$$\int_{\Omega} \nabla \cdot \phi \nabla G \, d\Omega = \int_S \phi \nabla G \cdot \hat{n} \, dS$$

→

$$\int_{\Omega} \nabla \phi \cdot \nabla G + \phi \nabla^2 G \, d\Omega = \int_S \phi \frac{\partial G}{\partial n} \, dS$$

Similarly, by taking $A = G \nabla \phi$ in the Gauss theorem, one can obtain

$$\int_{\Omega} \nabla G \cdot \nabla \phi + G \nabla^2 \phi \, d\Omega = \int_S G \frac{\partial \phi}{\partial n} \, dS$$

Recall that $\nabla^2 \phi = 0$ everywhere in the fluid domain and that $\nabla^2 G = G = 0$ everywhere except at the source point itself. Excluding the singular point by a small region of radius ϵ (we will consider the limit $\epsilon \rightarrow 0$ later) with boundary S_{ϵ} , we can then write the above two equations as

$$\begin{aligned} \int_{\Omega} \nabla \phi \cdot \nabla G \, d\Omega &= \int_S \phi \frac{\partial G}{\partial n} \, dS \\ \int_{\Omega} \nabla G \cdot \nabla \phi \, d\Omega &= \int_S G \frac{\partial \phi}{\partial n} \, dS \end{aligned}$$

where S is now $S = S_B + S_\epsilon + \Sigma$ which represent body surface, surface excluding singular point and far-field boundary, respectively. Subtracting the above two equations, as the left-hand side Ω integrals are the same, we get

$$\int_{S=S_B+S_\epsilon+\Sigma} \phi \frac{\partial G}{\partial n} dS = \int_{S=S_B+S_\epsilon+\Sigma} G \frac{\partial \phi}{\partial n} dS$$

Since, ϕ and $G \rightarrow 0$ at infinity, the integral for the far-field surface Σ can be dropped. We then have

$$\int_{S_B} \phi \frac{\partial G}{\partial n} dS + \int_{S_\epsilon} \phi \frac{\partial G}{\partial n} dS = \int_{S_B} G \frac{\partial \phi}{\partial n} dS + \int_{S_\epsilon} G \frac{\partial \phi}{\partial n} dS$$

For the three-dimensional case,

$$G = -\frac{1}{4\pi} \frac{1}{r}$$

Upon substitution, and cancellation of $-1/4\pi$ throughout, we get

$$\int_{S_B} \phi \frac{\partial}{\partial n} \frac{1}{r} dS + \int_{S_\epsilon} \phi \frac{\partial}{\partial n} \frac{1}{r} dS = \int_{S_B} \frac{1}{r} \frac{\partial \phi}{\partial n} dS + \int_{S_\epsilon} \frac{1}{r} \frac{\partial \phi}{\partial n} dS$$

where, recall that r is the distance between the source and field points:

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$$

In the above integral, with integration variable corresponding to $Q = Q(\xi, \eta, \zeta)$, r is then the distance from element dS to a field point $P = P(x, y, z)$. Note that $q \neq P$ because of region S_ϵ . To be sure that integration variables are (ξ, η, ζ) , let us re-write the above integral relation using Q :

$$\int_{S_B} \phi(Q) \frac{\partial}{\partial n_Q} \frac{1}{r} dS_Q + \int_{S_\epsilon} \phi(Q) \frac{\partial}{\partial n_Q} \frac{1}{r} dS_Q = \int_{S_B} \frac{1}{r} \frac{\partial \phi}{\partial n_Q} dS_Q + \int_{S_\epsilon} \frac{1}{r} \frac{\partial \phi}{\partial n_Q} dS_Q$$

The contributions of the integrals over S_ϵ can be evaluated analytically as follows:

$$\text{Lim}_{\epsilon \rightarrow 0} \int_{S_\epsilon} \phi(Q) \frac{\partial}{\partial n_Q} \frac{1}{r} dS_Q = \text{Lim}_{\epsilon \rightarrow 0} \phi(Q) (-) \frac{\partial}{\partial \epsilon} 4\pi\epsilon^2 = \text{Lim}_{\epsilon \rightarrow 0} \phi(Q) \frac{1}{\epsilon^2} 4\pi\epsilon^2 = 4\pi\phi(P)$$

In the above derivation, note the following

- $\frac{\partial}{\partial n} = -\frac{\partial}{\partial \epsilon}$ because \hat{n} is positive inward of S_ϵ (ie. out of Ω) and radial ϵ is positive outward;
- In the limit $\epsilon \rightarrow 0$, $Q \rightarrow P$.

- the point P is in Ω

The contribution of the other integral over S_ϵ is trivial:

$$\text{Lim}_{\epsilon \rightarrow 0} \int_{S_\epsilon} \frac{1}{r} \frac{\partial \phi}{\partial n_Q} dS_Q = \text{Lim}_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (-) \frac{\partial \phi}{\partial \epsilon} 4\pi\epsilon^2 = \text{Lim}_{\epsilon \rightarrow 0} \frac{\partial \phi}{\partial \epsilon} \frac{1}{\epsilon} 4\pi\epsilon^2 = 0$$

Thus the integral relation becomes, for $P \in \Omega$:

$$4\pi\phi(P) + \int_{S_B} \phi(Q) \frac{\partial}{\partial n_Q} \frac{1}{r} dS_Q = \int_{S_B} \frac{1}{r} \frac{\partial \phi}{\partial n_Q} dS_Q, \quad \text{for } P \in \Omega$$

One can similarly show that for $P \in S_B$ in which case the surface area of hemispherical S_ϵ is equal to $2\pi\epsilon^2$ the integral relation is

$$2\pi\phi(P) + \int_{S_B} \phi(Q) \frac{\partial}{\partial n_Q} \frac{1}{r} dS_Q = \int_{S_B} \frac{1}{r} \frac{\partial \phi}{\partial n_Q} dS_Q \quad \text{for } P \in S_B$$

The above integral relations are known as the Greens Theorem. They are also known as Greens Mixed Distribution as the right-hand side integral represent source ($\frac{1}{r}$) distribution and the integral on the left denote dipole ($\frac{\partial}{\partial n} \frac{1}{r}$) distribution.

In the case of two-dimensional flow, in which case

$$G = \frac{1}{2\pi} \ln r,$$

one can similarly obtain the Greens theorem as

$$-2\pi\phi(P) + \int_{S_B} \phi(Q) \frac{\partial}{\partial n_Q} \ln r dS_Q = \int_{S_B} \ln r \frac{\partial \phi}{\partial n_Q} dS_Q, \quad \text{for } P \in \Omega \text{ (2D)}$$

$$-\pi\phi(P) + \int_{S_B} \phi(Q) \frac{\partial}{\partial n_Q} \ln r dS_Q = \int_{S_B} \ln r \frac{\partial \phi}{\partial n_Q} dS_Q, \quad \text{for } P \in S_B \text{ (2D)}$$

Notice the minus (-) sign in the front of 2π and π terms.
