EOC6189 Computational Fluid Dynamics Fall 2006

Boundary Conditions for Equations Governing Incompressible Fluid Flow - Viscous Free-Surface Flow.

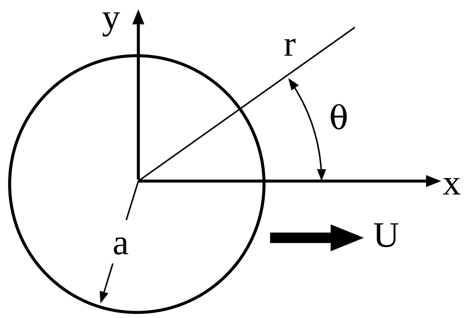
Modeling the open boundaries which can transmit waves without causing reflections and including effect of viscosity is not trivial. In simulating body-generated wave radiation problem, with initial condition corresponding to quiescent state, one can set the dynamic pressure field $p_{dynamic}$ to be zero on the lateral open boundary. As this is valid only until waves reach the open boundaries, the simulation has to be terminated before the waves reach the open boundaries.

A few other methods have been developed to model open boundaries such as coupling of inner viscous flow and outer inviscid flow, damping layer to dampen out the radiating waves before reaching open boundaries etc. We will examine these conditions in more detail later after discussing numerical methods for the field equations, ie. Laplace equation and Navier-Stokes equations.

Boundary-Integral Method for Potential Flow About an Arbitrary Body in ∞ Fluid.

Having reviewed formulation of incompressible fluid flows, let us examine methods to solve the flow equations. In this course, first we will discuss methods for potential flows with and without the presence of free surface and then methods for viscous flows. As you may recall from earlier wave-mechanics courses, there is a wide range of flow problems in which viscosity effect is only secondary, and therefore methods based on potential flow assumption still has practical value.

You may have already studied methods to solve the Laplace equation governing potential flow problem involving simple body geometry in earlier courses on applied mathematics and fluid mechanics. For example, let us consider translation of a 2D circular cylinder in infinite fluid as illustrated in the figure below.



In cylindrical polar coordinates (r, θ), the governing potential flow equations are

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$\frac{\partial \phi}{\partial r} = U \ \cos\theta \ \ \text{on r = a.}$$

$$|\nabla \phi| = 0 \text{ as } r \to \infty$$

The problem can be solved using the method of separation of variables (I'll leave this to you as home-work). The solution is given by

$$\phi = \frac{Ua^2}{r} \ \cos\theta$$

which is nothing but potential of a translating "dipole".

Such analytical methods will not work if the body geometry is arbitrary and complex as the one shown in the following figure.

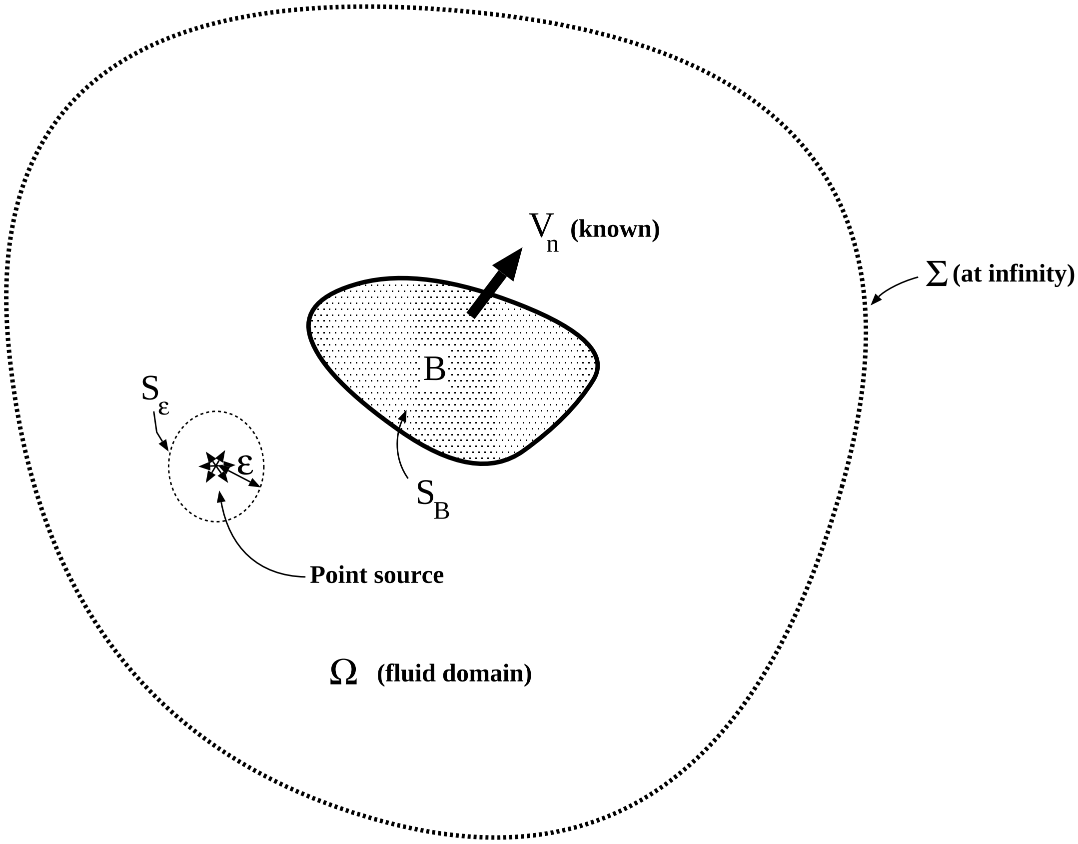
Here the body is moving with a specified normal velocity V_n. The governing equations are

$$\nabla^2 \phi = 0 \text{ in } \Omega$$

$$\frac{\partial \phi}{\partial n} = V_n \text{ on body surface } S_B$$

$$\phi = 0 \text{ at infinity}$$

Here the fluid domain is bounded by S_B and the far-field boundary Σ. The problem, which involves the arbitrary body boundary S_B, can be solved by boundary-integral method based on the Greens



function. The Greens function for this problem corresponds to potential due to a point source and is governed by

$$\nabla^2 G = \delta(P - Q)$$

which means $\nabla^2 G = 0$ everywhere except at the field point P = source point Q. Denoting the distance between field and source points as $r \equiv |P - Q|$, the above Laplace equation may be written as

$$\begin{aligned} \nabla^2 G &= \delta(r - 0) \\ &= 0 \text{ everywhere in } \Omega \text{ except at P=Q.} \end{aligned}$$

At infinity, $G \to 0$.

Because of the directional symmetry of the point source, the Laplace equation for the Greens function can be written (in spherical polar coordinates) as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{\partial G}{\partial r}\right) = 0, \text{ for } r > 0$$

Or, upon expanding as

$$\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} = 0, \ r > 0.$$

which can be solved easily as follow. Let

$$H \equiv \frac{dg}{dr}$$

which yields the following equation

$$\frac{dH}{dr} + \frac{2}{r}H = 0$$

solution of which is

$$H = \frac{C}{r^2}$$

where C is the integration constant. Integrating once more with respect to r:

$$H = \frac{dG}{dr} = \frac{C}{r^2} \ \rightarrow \ G = -\frac{C}{r} + D$$

where D is the second integration constant. As $G \rightarrow 0$ at infinity, $D = 0$. Thus,

$$G = -\frac{C}{r}, \text{ for } r > 0$$

The integration constant C can be found as follows. As the source considered is of unit strength,

$$\begin{aligned}
&\int_{r>0} \delta(r-0)\, d\Omega = 1,\\
&\rightarrow \int_{S_\epsilon} \nabla^2 G\, d\Omega = 1,\\
&\rightarrow \int_{S_\epsilon} \frac{\partial G}{\partial n}\, dS = 1\\
&\rightarrow \int_{S_\epsilon} \frac{\partial G}{\partial r}\, dS = 1\\
&\rightarrow \frac{C}{\epsilon^2}\, 4\pi\epsilon^2 = 4\pi C = 1\\
&\rightarrow C = \frac{1}{4\pi}
\end{aligned}$$

The 3D infinite-fluid potential-flow Green's function is thus

$$G = -\frac{1}{4\pi r}$$

which corresponds to potential due to a point source of unit strength at $r = 0$