Equations Governing Incompressible Fluid Flow

Let us next review the equations governing fluid flows, particularly of incompressibe fluids, introduced in earlier courses on fluid mechanics and hydrodynamics. As you may recall, these equations are derived based on the principles of classical physics and mathematical identities such as Gauss theorem and Reynolds Transport theorem. Here, we only list the equations without derivations.

Some standard notations used in the equations are as follows:

ρ = fluid density field

$\vec{u}$ = velocity field

p = pressure field

g = acceleration of gravity, taken to be acting in the negative z direction

$\vec{a} = \frac{D\vec{u}}{Dt}$ = fluid acceleration

$\frac{D}{Dt} = \frac{\partial u}{\partial t} + \vec{u} \cdot \nabla$ = total or material time derivative

μ = coefficient of dynamic viscosity

$\nu \equiv \mu/\rho$ = coefficient of kinematic viscosity

$\tilde{\sigma}$ = stress tensor

$\vec{\tau}$ = stress vector

$\hat{n}$ = unit normal (outward) vector

$\hat{t}$ = unit tangential vector

$\hat{b}$ = unit binormal vector

Ω = Volume of a control region

S = Boundary of a control region

δ_{ij} = Kronecker delta defined to be equal to 1 for i=j and zero for i $\neq$ j

Differential and integral forms of equations corresponding to the principle of conservation of mass are

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{u} = 0 \quad \rightarrow \nabla \cdot \vec{u} = 0,$$

$$\int_\Omega \frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{u} \, d\Omega = 0$$

respectively.

For incompressible fluid, these reduce to

$$\nabla \cdot \vec{u} = 0,$$

$$\int_\Omega \nabla \cdot \vec{u}\ d\Omega = \int_S \vec{u} \cdot \hat{n}\ dS = 0$$

The Euler's equation which correspond to the principle of balance of linear momentum of inviscid fluid is given by

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p - \rho g \hat{k}$$

The integral form of the above is given by

$$\int_\Omega \left(\frac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot (\rho \vec{u}) \vec{u} \right)\ d\Omega = -\int_S\ p\hat{n}\ dS - \int_\Omega\ \rho g \hat{k}\ d\Omega$$

In the case of an ideal flow, which corresponds to incompressible, inviscid and irrotational flow, the velocity field can be defined as the gradient of a scalar field ϕ known as the velocity potential; ie.

$$\nabla \times \vec{u} = 0 \quad \rightarrow \quad \vec{u} = \nabla \phi$$

The velocity potential is governed by the Laplace equation, as shown below.

$$\vec{u} = \nabla \phi \ \text{ in } \ \nabla \cdot \vec{u} = 0 \quad \rightarrow \quad \nabla^2 \phi = 0$$

By replacing u by $\nabla\phi$ in the Euler's equation and integrating in space, one can obtain the following Euler's integral which is also referred to as the unsteady Bernoulli's equation:

$$\vec{u} = \nabla \phi \ \text{ in the Euler's equation} \quad \rightarrow \quad p = -\rho(gz + \frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2)$$

As you may recall from earlier courses in solid or fluid mechanics, the following relation between stress vector $\vec{\tau}$ and stress tensor $\tilde{\sigma}$:

$$\vec{\tau} = \tilde{\sigma} \cdot \hat{n}$$

In the index notation, the above is written as

$$\tau_i = \sigma_{ij} n_j$$

where the repeated index j denotes summation.

The differential and integral forms of the principle of balance of lineam momentum (ie., Newton's II law) applied to real (viscous) fluids are given by

$$\rho \frac{D\vec{u}}{Dt} = +\nabla \cdot \tilde{\sigma} - \rho g \hat{k},$$

$$\int_\Omega \left(\frac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot (\rho \vec{u})\vec{u} \right) \, d\Omega = + \int_S \tilde{\sigma} \cdot \hat{n} \, dS - \int_\Omega \rho g \hat{k} \, d\Omega$$

respectively. In the case of a linear viscous (Newtonian), incompressible and homogeneous fluid, the constitutive relation is given by

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

where δ_{ij} denotes the Kronecker delta.

$$\begin{aligned} \delta_{i,j} &= 1, \text{ if } \; i = j \\ &= 0, \text{ if } \; i \neq j \end{aligned}$$

For such fluids, the equations governing the flow are given by the following incompressible Navier-Stokes Equations:

$$\nabla \cdot \vec{u} = 0$$

and

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p - \rho g \hat{k} + \mu \nabla^2 \vec{u}$$

where the unknowns are $\vec{u}$ and pressure p fields.

For later use for the development of boundary-element algorithms for inviscid flows, let us review some singularity solutions to the Laplace equation governing velocity potential. Derivation for 2D point source is given below.

<u>2D Simple Source at r = 0</u>

The velocity potential due to a point source at $r = 0$ is governed bu

$$\nabla^2 \phi = \delta(r - 0), \quad |\nabla \phi| \; \rightarrow \; 0 \text{ at infinity}$$

where δ denotes the delta function at the origin.

In cylindrical polar coordinates (flow is axisymmetric), the Laplace equation governing the flow in $r > 0$ is given by

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}=0,\ \text{for } r>0.$$

As $\phi=\phi(r)$, above can be written as

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}=0,\ \text{for } r>0.$$

which can be solved as follows:

Let $\psi=\frac{d\phi}{dr}$. The equation for ψ is then

$$\frac{d\psi}{dr}+\frac{1}{r}\psi=0,\ \text{for } r>0.$$

solution of which is

$$\ln\psi=-\ln r+\ln C^*,\ \text{where } C^* \text{ is a constant}$$

Or,

$$\psi=\frac{C}{r},\ \text{where } C \text{ is a constant, and for } r>0.$$

Integrating once more,

$$\phi=\int\ \psi dr\quad\rightarrow\quad\phi=C\ \ln r,\quad\text{for } r>0.$$

The integration constant C can be determined in terms of the source strength. Integrating the governing equation over a region around $r=0$:

$$\begin{aligned}
&\int\ \delta(r)\ d\Omega=\int\ \nabla^2\phi\ d\Omega\\
\rightarrow\ &=\int\ \nabla\cdot\nabla\phi\ d\Omega\\
\rightarrow\ &=\int\ \nabla\phi\cdot\hat{n}\ dS\ \text{(by Greens Theorem}\\
\rightarrow\ &=\int_{\theta=0}^{\theta=2\pi}\ \frac{d\phi}{dr}\ r\ d\theta\\
\rightarrow\ &=2\pi C
\end{aligned}$$

which represents the *flux* (ie. volume rate of flow) $\mathcal{F}$ of fluid from the source. Therefore,

$$C=\frac{\mathcal{F}}{2\pi}$$

and

$$\phi=\frac{\mathcal{F}}{2\pi}\ \ln r$$

Similarly, by solving (do this as home work) the Laplace equation in spherical polar coordinates, one can show that the potential due to a 3D source at the origin is given by

$$\phi=-\frac{\mathcal{F}}{4\pi}\ \frac{1}{r}$$