EOC6189 Computational Fluid Dynamics Fall 2006

ii. Diffusion Equation

Next, let us consider the one-dimensional diffusion equation given by

$$\frac{\partial u}{\partial t} = \nu \partial^2 u \partial x^2$$

where ν is a positive constant known as the diffusion coefficient. In view of simulations to be later considered in the course, let the boundary conditions be

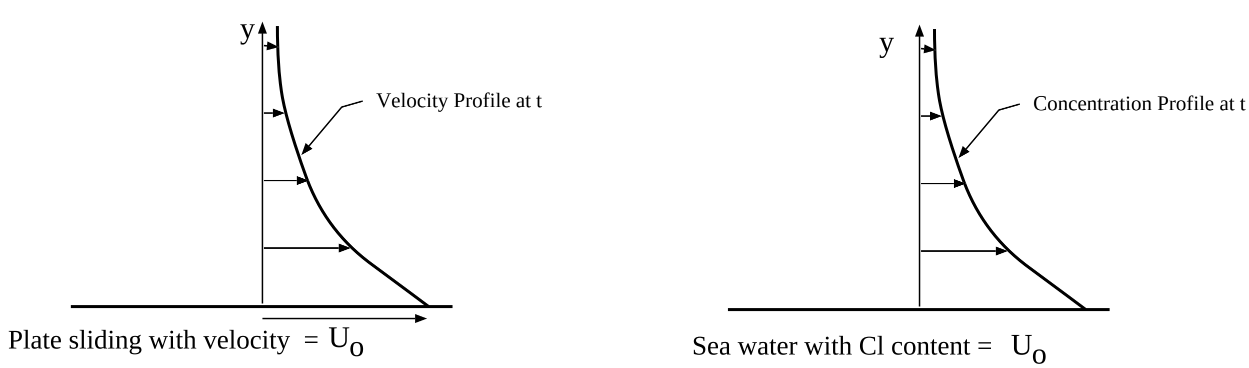
$$\begin{aligned} u(y = 0, t) &= 0, \quad \text{for } t < 0 \\ &= U_o, \quad \text{for } t \geq 0 \end{aligned}$$

and

$$u(y \to \infty, t) = 0$$

The above system of equations would represent the flow caused by an infinitely long plate (y=0) impulsively set into motion from time $t = 0$, as illustrated on the left of the figure. In this case u will correspond to horizontal flow velicity and U_o the velocity of the plate.

The equations could also represent diffusion of a substance brought into a contact with a medium; for example, diffusion of Cl ions into concrete exposed to seawater from time t = 0, as illustrated on the right of the figure. In this case, u will correspond to concentration (mass/volume) of the Cl and U_o the Cl concentration in sea water touching the concrete.



The above equation can be solved by similitude transformation. The similitude variable can be determined using the Buckingham's Pi theorem that you must have learnt in undergraduate Fluid Mechanics course. Let us then assume that u depends on y, t, ν, and U_o; ie.,

$$u = f(y, t, \nu, U_o)$$

which is a functional relation involving $N = 5$ variables. Of the five variables, letting briefly u to be anything (velocity, concentration, temperature, cat, whatever), there are at the most 3 variables (ie., M=3) which are dimensionally independent. For example, y, t and U_o is one such subset of dimensionally independent variables. Then by Pi theorem, the above functional relation can be replaced by a relation involving $N - M = 2$ non-dimensional variables:

$$\Pi_1 = F(\Pi_2)$$

where

$$\Pi_1 = \frac{u}{y^a t^b U_o^c} = \frac{u}{U_o}$$

and

$$\Pi_2 = \frac{\nu}{y^a t^b U_o^c} = \frac{\nu t}{y^2}$$

You may want to review the subject of Dimensional Analysis, in particular the Pi theorem and the exponent method, in case the above steps are not clear! The second non-dimensional variable can be also taken as

$$\Pi_2 = \frac{y}{\sqrt{\nu t}}$$

Thus by Pi theorem,

$$\frac{u}{U_o} = F(\frac{y}{\sqrt{\nu t}})$$

With $\frac{y}{\sqrt{\nu t}}$ taken as the similitude variable η, the above relation can be written as

$$\frac{u}{U_o} = F(\eta)$$

In other words the non-dimensional u/U_o depends only on one variable, namely the similitude variable η even though the dimensional u is function of y and t. This suggests the possibility of obtaining an ordinary differential equation for the non-dimensional u. As shown below, this is just a straightforward exercise involving chain-rule of differentiation.

As

$$\eta = \frac{y}{\sqrt{\nu t}}$$

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{d}{d\eta}\frac{\partial \eta}{\partial t} \\
&= -\frac{1}{2}\frac{y}{\sqrt{\nu}}\frac{1}{t^{3/2}}\frac{d}{d\eta} \\
&= -\frac{1}{2}\frac{y}{\sqrt{\nu t}}\frac{1}{t}\frac{d}{d\eta} \\
&= -\frac{1}{2}\eta\frac{1}{t}\frac{d}{d\eta}
\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial}{\partial y} &= \frac{d}{d\eta}\frac{\partial\eta}{\partial y} \\ &= \frac{1}{\sqrt{\nu t}}\frac{d}{d\eta}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial y^2} &= \frac{\partial}{\partial y}(\frac{1}{\sqrt{\nu t}}\frac{d}{d\eta}) \\ &= \frac{d}{d\eta}(\frac{1}{\sqrt{\nu t}}\frac{d}{d\eta})\frac{\partial\eta}{\partial y} \\ &= \frac{1}{\nu t}\frac{d^2}{d\eta^2}\end{aligned}$$

With the above transformation relations for derivatives, the diffusion equation

$$\frac{\partial u}{\partial t} = \nu\partial^2 u\partial x^2$$

becomes

$$-\frac{1}{2}\eta\frac{1}{t}\frac{du}{d\eta} = \nu\left[\frac{1}{\nu t}\frac{d^2u}{d\eta^2}\right]$$

Or,

$$\frac{d^2u}{d\eta^2} + \frac{1}{2}\eta\frac{du}{d\eta} = 0$$

Above equation can be solved as follows. Let

$$v \equiv \frac{du}{d\eta}$$

Then the above differential equation for u becomes,

$$\frac{dv}{d\eta} + \frac{1}{2}\eta v = 0$$

solution of which is

$$v = A\ e^{-\eta^2/4}$$

where A is constant of integration. Since $v = du/d\eta$,

$$u = \int^{\eta} A\ e^{-\alpha^2/4}\ d\alpha$$

$$u(y \to \infty, t) = 0$$

and because $\eta = \frac{y}{\sqrt{\nu t}}$

$$u = \int_{\infty}^{\eta} A\ e^{-\alpha^2/4}\ d\alpha$$

The integral corresponds to "error function" in statistics. The integral can be split as

$$
\begin{aligned}
u &= A\int_0^{\eta} e^{-\alpha^2/4}\, d\alpha - A\int_{\infty}^{0} e^{-\alpha^2/4}\, d\alpha \\
&= A\left[\int_0^{\eta} e^{-\alpha^2/4}\, d\alpha - \int_{\infty}^{0} e^{-\alpha^2/4}\, d\alpha\right]
\end{aligned}
$$

The second integral is equal to $\sqrt{\pi}$. Therefore

$$
u = A\left[\int_0^{\eta} e^{-\alpha^2/4}\, d\alpha - \sqrt{\pi}\right]
$$

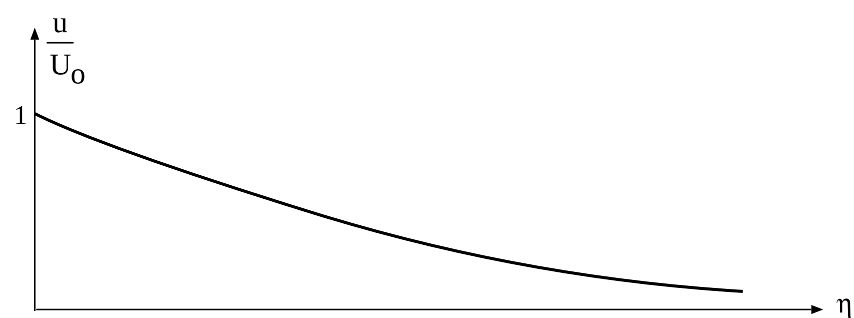
From the wall condition, $u = U_o$ on y (ie. η) $= 0$. Therefore

$$
A = -\frac{U_o}{\sqrt{\pi}}
$$

and thus the solution is

$$
\frac{u}{U_o} = \left[1 - \frac{1}{\sqrt{\pi}}\int_0^{\eta} e^{-\alpha^2/4}\, d\alpha\right]
$$

Graphical representation of the solution is shown below.



In fluid mechanics, the above problem also sheds light on viscous stress in impulsively started flows. Shear stress τ_{xy} is given, for this flow, by

$$
\tau_{xy} = \rho\nu\frac{\partial u}{\partial y}
$$

where ρ is fluid density. Using the solution obtained above,

$$
\begin{aligned}
\tau_{xy} &= \rho\nu\frac{du}{d\eta}\frac{\partial\eta}{\partial y} \\
&= \rho\nu\frac{1}{\sqrt{\nu t}}\frac{1}{\sqrt{\pi}}\, e^{-\eta^2/4}
\end{aligned}
$$

One can thus observe that viscous shear stress on the plate (ie. $\eta = 0$) is $1/\sqrt{t}$ singular in time. Later in the course, we will be discussing simulation of impulsively started motions in viscous fluid when the present result will help to understand small time numerical solutions better.