EOC6189 Computational Fluid Dynamics Fall 2006

Chapter 1. Exact Solutions to Model Partial Differential Equations

Let us begin with exact analysis of certain familiar partial differential equations that represent physical processes of fluid dynamics such as advection, diffusion and dispersion. The equations governing fluid dynamics problems are not so simple; nevertheless, understanding of the analysis and solution of model partial differential equation will enable one to efficiently develop or better understand algorithms for approximate solutions of fluid flow problems.

i. Advection Equation

A simple partial differential equation governing advection is given by

$$\frac{\partial u}{\partial t} + c\partial u\partial x =$$

where c, which represents the speed of advection, is a constant and $u \equiv u(x,t)$. Let the known initial value of u be

$$u(x, t = 0) = f(x)$$

As you may have studied in your earlier mathematics courses, this equation can be solved by a range of methods. For example, let us consider the one based on the Fourier transform method. Let

$$u^{(}*) = \frac{1}{\sqrt{2\pi}} \int u\ e^{-ikx}\ dx$$

$$u = \frac{1}{\sqrt{2\pi}} \int u^*\ e^{+ikx}\ dk$$

Substituting the Fourier integral representation of u in the given partial differential equation, one can obtain the following ordinary differential equation for the transform u^*:

$$\frac{du^*}{dt} + c(ik)u^* = 0$$

solution of which is

$$u^* = A\ e^{-ikct}$$

where A is the integration constant. At $t = 0$, $u = f(x)$ or $u^* = f^*$ where $f^*(k)$ is the Fourier transform of $f(x)$. Therefore, the integration constant $A = f^*$. Thus

$$u^* = f^*\ e^{-ikct}$$

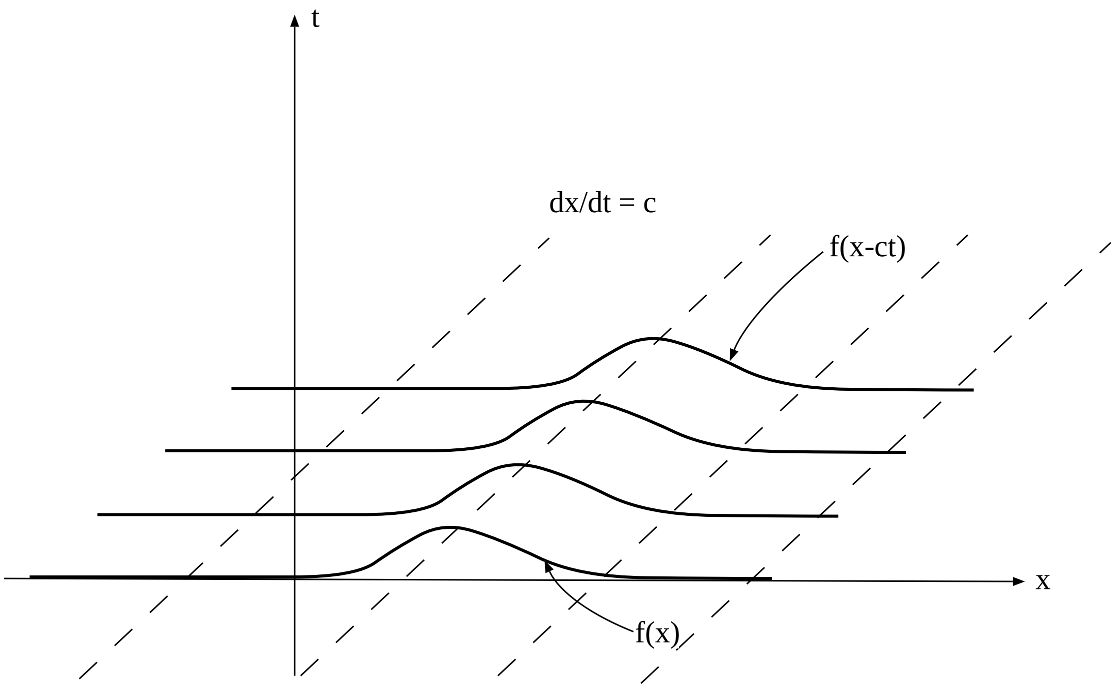
Substituting the above in the Fourier inverse transform, we get

$$\begin{aligned} u &= \frac{1}{\sqrt{2\pi}} \int u^*\ e^{+ikx}\ dk \\ &= frac1\sqrt{2\pi} \int f^*\ e^{-ikct}\ e^{+ikx}\ dk \\ &= \frac{1}{\sqrt{2\pi}} \int f^*\ e^{+ik(x-ct)}\ dk \\ &= f(x - ct) \end{aligned}$$

The solution of the partial differentail equation is thus

$$u(x,t) = f(x - ct)$$

The solution can be interpreted as follows. If the argument $(x - ct)$ is constant then u will be also constant. In other words, on lines on which $d(x - ct) = 0$, u will be consant. Lines on which solution u remains constant are called *characteristic* lines. The slope of the characteristic lines (on the characteristic t-x plane) is $\frac{dx}{dt} = c$. To an observer moving at $\frac{dx}{dt} = c$, u will appear to be stationary. Stated yet again differently, u is advected with velocity c! The solution for a typical initial value is illustrated in the following figure.



One could have also obtained the solution by assuming existence of the *characteristics* a priori. Since on the characteristics u is constant,

$$du(x,t) = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial t}\, dt = 0$$

Or slope of the characteristics is given by

$$\frac{dx}{dt} = -\frac{\partial u/\partial t}{\partial u/\partial x}$$

which by using the given partial differential equation can be written as

$$\frac{dx}{dt} = c$$

In other words, with respect to frame moving with $dx/dt = c$, u will appear to be stationary. That is, u is advected with speed c.
