Bernoulli's Equation and Euler's Integral

In the following, we will assume density ρ and acceleration of gravity g to be constants.

Case 1: Bernoulli's Equation for Steady Inviscid Flow. In this case, the equation of motion is given by the Euler's equation:

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right) = -\nabla p - \rho g\hat{k} = -\nabla(p + \rho g z)$$

where $\frac{\partial\vec{u}}{\partial t} = 0$ because of flow being steady. Therefore,

$$\rho(\vec{u}\cdot\nabla)\vec{u} = -(p + \rho g z)$$

The component of the above equation along a streamline direction s, (see figure below)

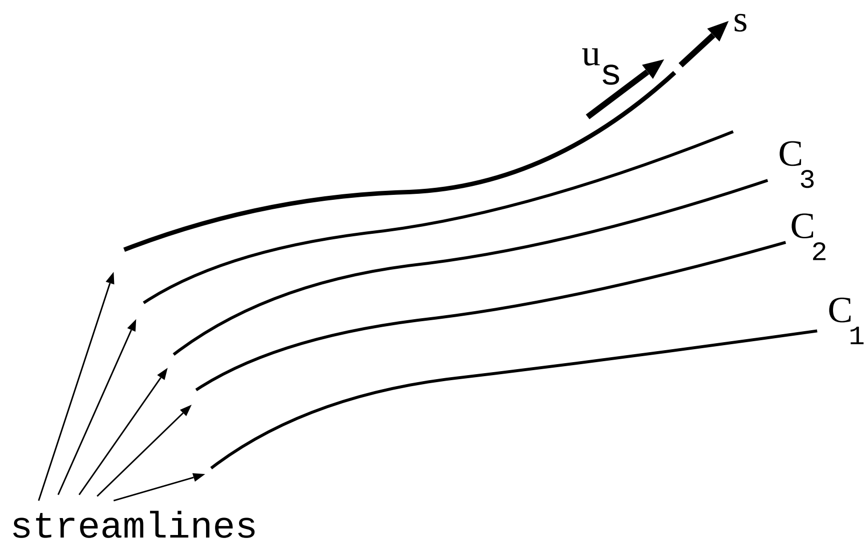

Fig. B-1 Steady Inviscid Flow

can be written as

$$\rho u_s \frac{\partial u_s}{\partial s} = -\frac{\partial}{\partial s}(p + \rho g z)$$

$\rightarrow$

$$\frac{\rho}{2}\frac{\partial u_s^2}{\partial s} = -\frac{\partial}{\partial s}(p + \rho g z)$$

$\rightarrow$

$$\frac{\partial}{\partial s}\left(\rho \frac{u_s^2}{2} + p + \rho g z\right) = 0$$

Since streamwise velocity u_s is the same as the magnitude of velocity U (flow is along streamlines),

$$\frac{\partial}{\partial s}\left(\rho \frac{U^2}{2} + p + \rho g z\right) = 0$$

Integrating and re-arranging terms,

$$p + \rho \frac{U^2}{2} + \rho g z = Constant, C \quad \text{along a streamline}$$

Note that the quantity on the left is constant only along the streamline. Different streamlines will have different constants, as denoted by C_1, C_2, C_3 etc in the figure on the previous page. Keep this in mind.

Case 2: Bernoulli's Equation for Steady Inviscid Irrotational Flow.

Next, let us consider the case in which the flow is not only steady but also irrotational. In this the Euler's equation of motion reduces as follows:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right) = -\nabla p - \rho g \hat{k} = -\nabla(p + \rho g z)$$

where $\frac{\partial \vec{u}}{\partial t} = 0$ because of flow being steady. Therefore,

$$\rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla(p + \rho g z)$$

Using vector identity

$$(\vec{u}\cdot\nabla)\vec{u} \equiv \frac{1}{2}\nabla|\vec{u}|^2 - (\nabla \times \vec{u}) \times \vec{u}$$

the above equation of motion can be written as

$$\rho\left(\frac{1}{2}\nabla|\vec{u}|^2 - (\nabla \times \vec{u}) \times \vec{u}\right) = -\nabla(p + \rho g z)$$

As the flow is also assumed to be irrotational (ie. $\nabla \times \vec{u} = 0$) the Euler's equation now reduces to

$$\rho \frac{1}{2} \nabla |\vec{u}|^2 = -\nabla (p + \rho g z)$$

Or,

$$\nabla \left(p + \frac{\rho}{2} |\vec{u}|^2 + \rho g z \right) = 0$$

Since the gradient ∇ (ie., in ALL directions) of the the quantities in the paranthesis is zero, and also the flow is steady,

$$p + \frac{\rho}{2} |\vec{u}|^2 + \rho g z = Constant \text{ (everywhere)}.$$

Denoting $|\vec{u}|$ as U, the equation can be written in a more familiar form as

$$p + \frac{\rho}{2} \frac{U^2}{2} + \rho g z = Constant$$

The difference between the above Bernoulli's equation for steady, inviscid and irrotational flow and the Bernoulli's equation given in Case 1 for steady inviscid flow is that in the present case (Case 2) the quantity

$$p + \frac{\rho}{2} \frac{U^2}{2} + \rho g z$$

is constant everywhere, whereas in Case 1, the quantity is constant only along a streamline. In Case 1, the constant will have different values on different streamlinesi as denoted by C_1, C_2 etc in the figure on the previous page. In Case 2, $C_1 = C_2 = C_3 = ... = C$. I hope difference in the Bernoulli's equation for steady flow and steady and irrotational flow is now clear.

Case 3: Euler's Integral a.k.a. Unsteady Bernoulli's Equation for the case of Unsteady Potential Flow.

Recall from lectures (refer to Lecture Notes #1), the basic equation governing an ideal flow (ie. incompressible inviscid irrotational) are given by

$$\vec{u} = -\nabla \phi$$

where ϕ is the velocity potential. Equation of continuity yields

$$\nabla \cdot \vec{U} = 0 \text{ (because of incompressibility)} \;\rightarrow\; \nabla^2 \phi = 0 \;\text{ (Laplace equation)}$$

Next, let us consider the substitution of $\vec{u} = -\nabla \phi$ in the Euler's equation. For this, we consider the following version of the Euler's equation:

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \frac{1}{2} \nabla |\vec{u}|^2 - (\nabla \times \vec{u}) \times \vec{u} \right) = -\nabla (p + \rho g z)$$

Note that the second and third terms, by vector identity, are equal to $\vec{u} \cdot \nabla \vec{u}$. Becauase of irrotationality $\nabla \times \vec{u} = 0$; the above Euler's equation therefore becomes

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \frac{1}{2} \nabla |\vec{u}|^2 \right) = -\nabla (p + \rho g z)$$

Substituting $\vec{u} = -\nabla \phi$ and combining terms, we get

$$\nabla \left(-\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + \frac{p}{\rho} + gz \right) = 0$$

As the gradient of the quantities in the paranthesis is equal to zero, and as the flow is NOT steady, above implies

$$\left(-\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + \frac{p}{\rho} + gz \right) = C(t)$$

which is known as (correctly) Euler's integral and (incorrectly) as Bernoulli's equation! Note here C is function of time. Recall that in cases 1 and 2, the Bernoulli's constant C is not function of time. In the Bernoull's equation of Case 1, C is function of streamlines, in the Bernoulli's equation of Case 2, C is same everywhere in the fluid. Those cases (1 and 2) are for steady flows. In the present case of unsteady irrotational flow, $C = C(t)$.

This time function $C(t)$ can be absorbed in ϕ in the following manner. Let

$$-\frac{\partial \phi}{\partial t} - C(t) \equiv -\frac{\partial \phi^*}{\partial t}$$

where ϕ^* is a "new" potential. Integrating with respect to time,

$$\phi^* = \phi + \int^t C(\tau) \, d\tau$$

Since ϕ^* and ϕ differ only by a time function,

$$\nabla \phi^* = \nabla \phi = -\vec{u}$$

In other words, both ϕ and new ϕ^* give the same flow velocity! The Euler's integral can now be written as

$$\left(-\frac{\partial \phi^*}{\partial t} + \frac{1}{2} |\nabla \phi^*|^2 + \frac{p}{\rho} + gz \right) = 0$$

In this form of Euler's integral, the right-hand side is zero! The question now is, what are the equations governing ϕ^* ? As observed above, as

$$\nabla \phi = \nabla \phi^* = -\vec{u}$$

Therefore, because of incompressibility,

$$\nabla \cdot \vec{u} = 0 \quad \rightarrow \nabla^2 \phi^* = 0$$

In other words, ϕ^* is also governed by the same Laplace equations. In sum, the equations of flow motion, with the introduction of a new potential ϕ^*, are given by

$$\vec{u} = -\nabla \phi^*$$

$$\nabla^2 \phi^* = 0$$

$$\left(-\frac{\partial \phi^*}{\partial t} + \frac{1}{2} |\nabla \phi^*|^2 + \frac{p}{\rho} + gz \right) = 0$$

which are basically same as that for ϕ except with the use of ϕ^* the time function $C(t)$ appearing in the Euler's integral has disappeared. Hitherto, we will define the potential flow with ϕ^* and for convenience drop the $*$ superscript.

Absorption of $C(t)$ is not so possible if the flow is NOT unsteady! Let us go back to the Euler's integral with $C(t)$, given earlier as,

$$\left(-\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + \frac{p}{\rho} + gz \right) = C(t)$$

If the flow is steady, then $\frac{\partial \phi}{\partial t} = 0$ and $C(t) = C$ (where is C constant). Then

$$\frac{1}{2} |\nabla \phi|^2 + \frac{p}{\rho} + gz = C$$

Observing $|\nabla \phi|^2 = U^2$, above can be written as

$$\frac{1}{2} U^2 + \frac{p}{\rho} + gz = C \text{ (constant everywhere in the flow)}$$

which is same as Case 2 Bernoulli's equation. Here there is no way of getting rid of the constant C as we did before in Case 3 for the Euler's integral! Of course, we can redefine pressure, to get rid of C; for example as

$$\frac{p}{\rho} - C = \frac{p^*}{\rho}$$

in which case the above equation will reduce to

$$\frac{1}{2} U^2 + \frac{p^*}{\rho} + gz = 0$$

but on the free surface the new pressure p^* will not be equal to p_{atm}! Wave Mechanics is an interesting subject or what?!
